Forbidden pairs of disconnected graphs for 2-factor of connected graphs

Přemysl Holub [1,4,7] Zdeněk Ryjáček [1,4,7] Petr Vrána [1,4,7]
Shipeng Wang [2,5,8] Liming Xiong [3,6,9]

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Abstract

Let $\mathcal{H}$ be a set of graphs. A graph G is said to be $\mathcal{H}$-free if G does not contain H as an induced subgraph for all H in $\mathcal{H}$, and we call $\mathcal{H}$ a forbidden pair if $|\mathcal{H}| = 2$. Faudree et al. (2008) characterized all pairs of connected graphs R, S such that every 2-connected $\{R, S\}$-free graph of sufficiently large order has a 2-factor. In 2013, Fujisawa et al. characterized all pairs of connected graphs R, S such that every connected $\{R, S\}$-free graph of sufficiently large order with minimum degree at least two has a 2-factor.

In this paper, we generalize these two results by considering disconnected graphs R, S. In other words, we characterize all pairs of graphs R, S such that every 2-connected $\{R, S\}$-free graph of sufficiently large order has a 2-factor. We also characterize all pairs of graphs R, S such that every connected $\{R, S\}$-free graph of sufficiently large order with minimum degree at least two has a 2-factor.

Keywords: forbidden subgraph; disconnected graph; 2-factor; closure

1 Introduction.

We basically follow the most common graph-theoretical terminology and notation and for concepts not defined here we refer the reader to [2]. All graphs in this paper are simple, finite and undirected.

[1]European Centre of Excellence NTIS, Department of Mathematics, University of West Bohemia, Univerzitní 8, 301 00 Pilsen, Czech Republic

[2]Institute of Applied System Analysis, Jiangsu University, Zhenjiang, Jiangsu, 212013, China.

[3]School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China

[4]e-mail {holubpre,ryjacek,vranap}@kma.zcu.cz

[5]e-mail spwang22@yahoo.com

[6]e-mail lmxiong@bit.edu.cn

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Let G be a graph, $u, v \in V(G)$, $X \subseteq V(G)$, and let H be a subgraph of G. Then $N_G(v)$ denotes the set, and $d_G(v)$ the number, of neighbors of v in G, $d_H(v)$ the number of neighbors of v in H, $N_G(X)$ the set of vertices of $V(G) \setminus X$ having a neighbor in X, and $N_H(X)$ the set of vertices of $V(H) \setminus X$ having a neighbor in X. We use $n(G)$ to denote the order of G, $e(G)$ the size of G, $\alpha(G)$ the independence number of G, $\kappa(G)$ the connectivity of G and $\mathrm{nc}(G)$ the number of components of G. By a *clique* in G we mean a complete subgraph of G (not necessarily maximal). A *pendant vertex* is a vertex of degree 1, and a *pendant edge* is an edge incident with a pendant vertex. The distance between u and v in G is denoted $\mathrm{dist}_G(u, v)$, and, when $u, v \in V(H)$, $\mathrm{dist}_H(u, v)$ denotes their distance in the subgraph H of G, i.e., the length of a shortest path between u and v in H. A path joining vertices u and v will be called a *(u, v)-path*, and, analogously, for vertex subsets $X, Y \subseteq V(G)$, an (X, Y)-path is a path with one endvertex in X and the other endvertex in Y. We also use E_x to denote the set of edges between x and all its neighbors.

For $X \subset V(G)$ (or $X \subset E(G)$), $\langle X \rangle_G$ denotes the subgraph of G induced by the set of vertices X (or determined by the set of edges X) in G, respectively. A graph G is called *H-free* if G does not contain H as an induced subgraph. Analogously, for a set $\mathcal{H}$ of graphs, G is called *$\mathcal{H}$-free* if G does not contain any graph from $\mathcal{H}$ as an induced subgraph. In this context it is common to call such a graph H (or a member of a class $\mathcal{H}$) a *forbidden subgraph*. We use $H_1 \cup H_2$ to denote the *disjoint union* of two vertex-disjoint graphs H_1 and H_2. Thus, $(H_1 \cup H_2)$-free means to forbid $H_1 \cup H_2$ as an induced subgraph, it does not mean forbidding H_1 and/or H_2.

We will use the following notations for some special graphs: K_i ($i \geq 1$) - the complete graph on i vertices, $K_{1,r}$ ($r \geq 2$) - a star, P_i ($i \geq 1$) - the path on i vertices (so $P_1 = K_1, P_2 = K_2$). We use $N_{i,j,k}$ to denote the graph obtained by attaching three vertex-disjoint paths of lengths $i, j, k \geq 0$ to a triangle. In the special case when $i, j \geq 1$ and $k = 0$ (or $i \geq 1$ and $j = k = 0$), $N_{i,j,k}$ is also denoted $B_{i,j}$ (or Z_i), respectively (see Fig. $1(a), (b), (c)$). We use L_i ($i \geq 2$) to denote the graph obtained from K_i by adding a pendant edge (so $L_2 = P_3$ and $L_3 = Z_1$).

The *Ramsey number $R(k, l)$* is defined as the smallest integer n such that every graph on n vertices contains either a clique on k vertices or an independent set of l vertices. A graph G is called *hamiltonian*, if it contains a Hamilton cycle, i.e., a cycle containing all vertices of G. A path in G containing all vertices of G is called a *Hamilton path*. A graph G is called *Hamilton-connected* if it contains a Hamilton (x, y)-path for each pair x, y of vertices of G. A *2-factor* of a graph is a spanning subgraph whose components are cycles. A graph is called *2-factorable* if it contains a 2-factor. The *Theta graph* $\Theta(i, j, k)$ consists of a pair of endvertices joined by three internally disjoint paths of lengths $i + 1, j + 1, k + 1$, $i \geq j \geq k \geq 1$ (see Fig. $1(d)$). Unless otherwise stated, we will always keep the notation of vertices of a $\Theta(i, j, k)$ as in Fig. $1(d)$.

The first characterization of forbidden pairs of connected subgraphs for hamiltonicity of 2-connected graphs was given by Bedrossian in [1].

Theorem A [1]. *Let R, S be a pair of connected graphs such that neither R nor S is an induced subgraph of P_3. Then G being a 2-connected $\{R, S\}$-free graph implies that G is*

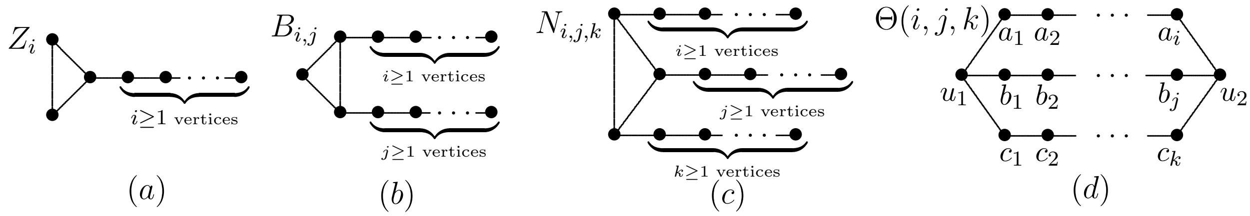

Figure 1: The graphs Z_i, $B_{i,j}$, $N_{i,j,k}$ and $\Theta(i,j,k)$

hamiltonian if and only if (up to a symmetry), $R = K_{1,3}$ and S is an induced subgraph of $P_6, B_{1,2}$ or $N_{1,1,1}$.

Faudree and Gould [6] observed that there are only finitely many nonhamiltonian $\{K_{1,3}, Z_3\}$-free graphs, which implies the following improvement of Theorem A.

Theorem B [6]. *Let R, S be a pair of connected graphs such that neither R nor S is an induced subgraph of P_3. Then every 2-connected $\{R, S\}$-free graph of order at least 10 is hamiltonian if and only if (up to a symmetry), $R = K_{1,3}$ and S is an induced subgraph of $P_6, B_{1,2}, N_{1,1,1}$ or Z_3.*

Faudree et al. [7] characterized all forbidden pairs of connected subgraphs for 2-factor of 2-connected graphs of sufficiently large order.

Theorem C [7]. *Let R, S be a pair of connected graphs such that neither R nor S is an induced subgraph of P_3. Then every 2-connected $\{R, S\}$-free graph of order at least 10 has a 2-factor if and only if (up to a symmetry), $R = K_{1,3}$ and S is an induced subgraph of $P_7, B_{1,4}, N_{1,1,3}$, or $R = K_{1,4}$ and $S = P_4$.*

An analogous result for connected graphs with minimum degree 2 was given by Fujisawa and Saito [8].

Theorem D [8]. *Let R, S be a pair of connected graphs such that neither R nor S is an induced subgraph of P_3. Then there exists a positive integer n_0 such that every connected $\{R, S\}$-free graph of order at least n_0 and minimum degree at least two has a 2-factor if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of Z_2.*

Li and Vrána [11] extended Theorem B by considering disconnected graphs R, S.

Theorem E [11]. *Let R, S be a pair of graphs such that neither R nor S is an induced subgraph of P_3 or $3K_1$. Then there exists a positive integer n_0 such that every 2-connected $\{R, S\}$-free graph of order at least n_0 is hamiltonian, if and only if (up to a symmetry):*

(i) $R = K_{1,3}$ and S is an induced subgraph of P_6, Z_3, $B_{1,2}$, $N_{1,1,1}$, $K_1 \cup Z_2$, $K_2 \cup Z_1$, or $K_3 \cup P_4$;

(ii) $R = K_{1,k}$ with $k \geq 4$ and S is an induced subgraph of $2K_1 \cup K_2$;

(*iii*) $R = kK_1$ *with* $k \geq 4$ *and* S *is an induced subgraph of* L_l *with* $l \geq 3$*, or* $2K_1 \cup K_l$ *with* $l \geq 2$.

In this paper, we extend Theorems C and D in a similar way as Theorem E extends Theorem B. Proofs of Theorems 1, 2 and 3 are postponed to Section 4.

Our first result characterizes all (possibly disconnected) graphs F such that every "sufficiently large" 2-connected F-free graph has a 2-factor.

Theorem 1. *Let* F *be a graph. Then* G *being 2-connected* F*-free of order at least* $R(31, 4)$ *implies* G *has a 2-factor if and only if* F *is an induced subgraph of* P_3 *or* $4K_1$*.*

When forbidding a pair of graphs R, S such that every 2-connected $\{R, S\}$-free graph (of sufficiently large order) has a 2-factor, to avoid trivial cases, we suppose that neither R nor S is an induced subgraph of P_3 or $4K_1$ by virtue of Theorem 1. The following theorem can be considered as a generalization of Theorem C.

Theorem 2. *Let* R, S *be a pair of graphs such that neither* R *nor* S *is an induced subgraph of* P_3 *or* $4K_1$*. Then there exists a positive integer* n_0 *such that every 2-connected* $\{R, S\}$*-free graph of order at least* n_0 *has a 2-factor if and only if (up to a symmetry):*

(*i*) $R = K_{1,3}$ *and* S *is an induced subgraph of* $P_7, B_{1,4}, N_{1,1,3}, K_3 \cup Z_1, Z_1 \cup P_4, Z_4 \cup K_1, N_{1,1,1} \cup K_2$*, or* $K_3 \cup P_4 \cup K_1$*;*

(*ii*) $R = K_{1,4}$ *and* S *is an induced subgraph of* $P_3 \cup 2K_1$*, or* $3K_1 \cup K_2$*;*

(*iii*) $R = K_{1,k}$ *with* $k \geq 5$ *and* S *is an induced subgraph of* $3K_1 \cup K_2$*;*

(*iv*) $R = kK_1$ *with* $k \geq 5$ *and* S *is an induced subgraph of* L_l *with* $l \geq 3$*, or* $3K_1 \cup K_l$ *with* $l \geq 2$.

In [8], Fujisawa and Saito proved the following.

Theorem F [8]. *Let* G *be a connected graph order at least 6, independence number* $\alpha(G) \leq 2$ *and minimum degree at least two. Then* G *has a 2-factor.*

Similarly as in Theorem 2, to avoid trivial cases, our next main result requires that neither R nor S is an induced subgraph of P_3 or $3K_1$ (by virtue of Theorem F).

Theorem 3. *Let* R, S *be a pair of graphs such that neither* R *nor* S *is an induced subgraph of* P_3 *or* $3K_1$*. Then there exists a positive integer* n_0 *such that every connected* $\{R, S\}$*-free graph of order at least* n_0 *and minimum degree at least two has a 2-factor if and only if (up to a symmetry):*

(*i*) $R = K_{1,3}$ *and* S *is an induced subgraph of* $Z_2, P_3 \cup K_2, Z_1 \cup K_2$ *or* $K_1 \cup K_2 \cup K_3$*;*

(*ii*) $R = K_{1,k}$ *with* $k \geq 4$ *and* S *is an induced subgraph of* $2K_1 \cup K_2$*;*

(*iii*) $R = 4K_1$ *and* S *is an induced subgraph of* L_l *with* $l \geq 3$*, or* $K_1 \cup K_2 \cup K_l$ *with* $l \geq 2$*;*

(*iv*) $R = kK_1$ *with* $k \geq 5$ *and* S *is an induced subgraph of* L_l *with* $l \geq 3$*, or* $2K_1 \cup K_l$ *with* $l \geq 2$.

In the next section, we will present some necessary results on line graphs and on the closure operation for claw-free graphs, and some further known results that will be needed. In Section 3, we collect partial results that will compose sufficiency parts of the proofs of Theorems 2 and 3. Finally, in Section 4, we complete the proofs of the main results.

2 Preliminaries

The line graph of a graph H, denoted $L(H)$, has $E(H)$ as its vertex set, where two vertices are adjacent in $L(H)$ if and only if the corresponding edges of H have a vertex in common. It is a well-known fact that if G is a connected line graph different from K_3, then the graph H such that $L(H) = G$, is uniquely determined. This graph will be called the *preimage* of G, and denoted $L^{-1}(G)$. A graph is *essentially k-edge-connected* if every edge cut of size less than k is trivial (no more than one component of the graph after deleting the edge cut contains any edges). It is easy to see that G is k-connected if and only if $L^{-1}(G)$ is essentially k-edge-connected.

Ryjáček [13] introduced the closure of a claw-free graph, which became a useful tool for investigation of hamiltonian properties of claw-free graphs. A vertex $x \in V(G)$ is said to be *eligible* if $\langle N_G(x)\rangle$ is a connected non-complete graph. We will use $V_{EL}(G)$ to denote the set of all eligible vertices of G. For $x \in V_{EL}(G)$, the graph G'_x obtained from G by adding the edges $\{yz : y, z \in N_G(x) \text{ and } yz \notin E(G)\}$ is called the *local completion* of G at x. The *closure* of a claw-free graph G is the graph $\mathrm{cl}(G)$ obtained from G by recursive performing the local completion operation at eligible vertices, as long as this is possible (more precisely, there is a sequence of graphs $G_1, \cdots, G_k$ such that $G_1 = G$, $G_{i+1} = (G_i)'_x$ for some vertex $x \in V_{EL}(G_i)$, $i = 1, \ldots, k-1$, and $G_k = \mathrm{cl}(G)$. The following theorem provides fundamental properties of the closure operation.

Theorem G [13]. *Let G be a claw-free graph. Then*
(*i*) $\mathrm{cl}(G)$ *is uniquely determined;*
(*ii*) $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph;*
(*iii*) *G is hamiltonian if and only if* $\mathrm{cl}(G)$ *is hamiltonian.*

Following [3], we say a class $\mathcal{H}$ of graphs is *stable* under the closure if, for every $G \in \mathcal{H}$, cl(G) is also in $\mathcal{H}$. Ryjáček et al. [14] proved that the property of a claw-free graph having a 2-factor is stable under the closure.

Theorem H [14]. *Let G be a claw-free graph. Then G has a 2-factor if and only if* cl(G) *has a 2-factor.*

Brousek et al. [3] showed stability of some classes of graphs defined in terms of forbidden pairs.

Theorem I [3]. *Let S be a connected graph of order at least 3. If $S \in \{K_3\} \cup \{Z_i : i > 0\} \cup \{N_{i,j,k} : i, j, k > 0\}$, then the class of $\{K_{1,3}, S\}$-free graphs is stable under the closure.*

Later, Li and Vrána considered the analogue of Theorem I for disconnected graphs.

Theorem J [11]. *Let S be a disconnected graph of order at least 3. Then the class of $\{K_{1,3}, S\}$-free graphs is stable, if and only if, for every component C of S, the class of $\{K_{1,3}, C\}$-free graphs is stable.*

Brousek et al. [3] showed that the class of $\{K_{1,3}, B_{i,j}\}$ $(i, j \geq 1)$-free graphs is not stable. Recently, Du and Xiong considered the stability of $\{K_{1,3}, B_{i,j}\}$ $(i, j \geq 1)$-free graphs with three pendant vertices.

Theorem K [5]. *Let G be a connected claw-free graph with three pendant vertices v_1, v_2, v_3. Then for any pair of $v_i, v_j \in \{v_1, v_2, v_3\}$, G has an induced subgraph $B_{l,k}$ containing v_i, v_j for some $l, k \geq 1$.*

Let F be a subgraph of a graph H. We say that F is *dominating* in H if every edge of H has at least one end in F, and that F is *even* if every vertex of F has even degree in F. A set $\mathcal{D}$ of even subgraphs and stars with at least three edges in H is called a *d-system* of H, if every edge of H is contained in a member of $\mathcal{D}$ or incident with a vertex in an even subgraph in $\mathcal{D}$. Harary and Nash-Williams [10] showed that for a graph H with $|E(H)| \geq 3$, $L(H)$ is hamiltonian if and only if H has a dominating connected even subgraph. A similar relation between a 2-factor in a line graph G and a d-system in its preimage $L^{-1}(G)$ was established by Gould and Hynds [9].

Theorem L [9]. *Let H be a graph with $|E(H)| \geq 3$. Then $L(H)$ has a 2-factor if and only if H has a d-system.*

We further list here some classical results which will be used for the proof of the main results of this paper.

Theorem M (Mantel) [12]. *Every K_3-free graph of order n has at most $n^2/4$ edges.*

Theorem N (Chvátal and Erdős) [4]. *Let G be a graph on at least three vertices with independence number α and connectivity κ. If $\alpha \leq \kappa$ (or $\alpha \leq \kappa - 1$), then G is hamiltonian (or Hamilton-connected), respectively.*

The following result for 2-connected graphs is implicit in the proof of the main result of [11]. Since it is actually true for connected graphs, we present its proof here.

Theorem O [11]. *Every connected $\{kK_1, L_l\}$-free graph, $k, l \geq 3$, of order at least $R(2l-3, k) + k - 2$ is hamiltonian.*

Proof. Since G is kK_1-free, we have $\alpha(G) \leq k-1$. If $\kappa(G) \geq k-1$, then G is hamiltonian by Theorem N. Hence we assume that $\kappa(G) \leq k-2$. Let S be a smallest vertex cut of G. Then $|S| \leq k-2$. Since $G-S$ is kK_1-free and $n(G) \geq R(2l-3,k)+k-2$, $G-S$ contains a clique T of order $2l-3$. Let v_1 be a vertex of $G-S$ such that v_1 and T are in distinct components of $G-S$. Then v_1 has no neighbor in T. Let $P = v_1v_2\cdots v_p$ be a shortest (v_1,T)-path. Then the length of P is at least two, i.e., $p \geq 3$. Let us consider the neighborhood of v_{p-1} in T. If v_{p-1} has at least $l-1$ neighbors in T, then $\langle N_T(v_{p-1}) \cup \{v_{p-2}\}\rangle_G$ contains an induced L_l, a contradiction. Hence we assume that v_{p-1} has at most $l-2$ neighbors in T. Then $|V(T) \setminus N_T(v_{p-1})| \geq l-1$ since $|V(T)| = 2l-3$, thus $\langle V(T) \setminus N_T(v_{p-1}) \cup \{v_p, v_{p-1}\}\rangle_G$ contains an induced L_l, a contradiction. ■

Theorem P [8]. *Let G be a connected $\{K_{1,3}, Z_2\}$-free graph with minimum degree at least two. Then G is hamiltonian or G is isomorphic to $K_1 + (K_l \cup K_m)$ for some integers l and m with $l, m \geq 2$.*

This theorem immediately implies the following consequence.

Corollary 4. *Every connected $\{K_{1,3}, Z_1\}$-free graph with minimum degree at least two is hamiltonian.*

Lemma 5. *Let G be a 2-connected non-2-factorable line graph. Then the graph $H = L^{-1}(G)$ contains a subgraph isomorphic to $\Theta(k_1,k_2,k_3)$ with $k_1,k_2,k_3 \geq 2$.*

Proof. Let T be the set of all pendant vertices of H. If there is a cut-edge e in $H-T$, then $H-e$ has two nontrivial components, implying that G is not 2-connected, a contradiction. Hence $H-T$ is 2-edge-connected. Let $B'_1,\ldots,B'_t$ be all blocks of $H-T$ and let $\mathcal{F} := \{B_1,\ldots,B_t\}$ be a decomposition of H such that $B_i \cap (H-T) = B'_i$, $i = 1,\ldots,t$. For $1 \leq i \leq t$, each B'_i contains a cycle since $H-T$ is 2-edge-connected.

If each $L(B_i)$ has a 2-factor $\mathcal{C}_i$, $i = 1,\ldots,t$, then $\mathcal{C}_1 \cup \ldots \cup \mathcal{C}_t$ is a 2-factor of G, a contradiction. Hence there exists a $B_i \in \mathcal{F}$, say, B_1, such that $L(B_i)$ has no 2-factor. Then B_1 has no d-system by Theorem L. Let C be a longest cycle of B'_1. Suppose that each component of $B'_1 - V(C)$ is trivial (having one vertex only), and let $v_1,\ldots,v_s$ denote all components (i.e., vertices) of $B'_1 - V(C)$ that have a neighbor in T. Since H is essentially 2-edge-connected, each of $v_1,\ldots,v_s$ has at least two neighbors on C, and, since C is longest, these two neighbors are not consecutive on C. Then C together with the stars $E_{v_1},\ldots,E_{v_s}$ is a d-system of B_1, a contradiction. Therefore there is some nontrivial component of $B'_1 - V(C)$; let D denote such a component. Then D contains a nontrivial path P in D with endvertices denoted x,y, $x \neq y$. Since B'_1 is 2-connected, there is a pair of vertices $u,v \in V(C)$ such that $xu, yv \in E(B_1)$. Since C is a longest cycle of B'_1, $\mathrm{dist}_C(u,v) \geq 3$, implying that $|V(C)| \geq 6$. Hence $\langle V(C) \cup V(P)\rangle_H$ contains a subgraph isomorphic to $\Theta(k_1,k_2,k_3)$ with $k_1,k_2,k_3 \geq 2$. ■

Lemma 6. *Let G be a 2-connected kK_1-free graph, $k \geq 2$, such that $V(G)$ can be partitioned into two sets X and Y satisfying the following:*

(*i*) *$\langle X\rangle_G$ contains a clique T such that every vertex of X has at least $k+7$ neighbors in T;*

(*ii*) *$\alpha(\langle Y\rangle_G) \leq 2$.*

Then G has a 2-factor.

Proof. We start with the following fact.

Claim 1. *For any set $X' \subset X$ with $|X'| \leq 8$, $\langle X \setminus X'\rangle_G$ is hamiltonian.*

Proof. By (*i*), we have $\kappa(\langle X\rangle_G) \geq k+7$. Since G is kK_1-free, $\alpha(\langle X\rangle_G) \leq \alpha(G) \leq k-1$. For any set $X' \subset X$ with $|X'| \leq 8$, we have $\alpha(\langle X \setminus X'\rangle_G) \leq \alpha(\langle X\rangle_G) \leq k-1$ and $\kappa(\langle X \setminus X'\rangle_G) \geq \kappa(\langle X\rangle_G) - 8 \geq k+7-8 = k-1$. By Theorem N, $\langle X \setminus X'\rangle_G$ is hamiltonian. $\square$

For $Y = \emptyset$, G is hamiltonian by Claim 1. Hence we assume that $Y \neq \emptyset$. If $\kappa(\langle Y\rangle_G) \geq 2$, then by (*ii*), $\alpha(\langle Y\rangle_G) \leq \kappa(\langle Y\rangle_G)$ and hence $\langle Y\rangle_G$ is hamiltonian, implying that G has a 2-factor with exactly two components by Claim 1. Hence we assume that $\kappa(\langle Y\rangle_G) \leq 1$. We now consider the following two cases.

Case 1: $\kappa(\langle Y\rangle_G) = 1$.

Let v be a cut-vertex in $\langle Y\rangle_G$. By (*ii*), $\langle Y\rangle_G - v$ has exactly two components D_1 and D_2 such that each of D_1 and D_2 is a clique. Since G is 2-connected, there exist two edges between $V(D_1 \cup D_2)$ and X, say $v_1x_1, v_2x_2 \in E(G)$, with $x_1, x_2 \in X$, $x_1 \neq x_2$, and $v_1 \in V(D_1)$, $v_2 \in V(D_2)$, $v_1 \neq v_2$. Since both D_1 and D_2 are cliques, there is a Hamilton (v_1, v_2)-path P of $\langle Y\rangle_G$. By the definition of X, there is an edge x_3x_4 in T such that $x_1x_3, x_4x_2 \in E(G)$, and then $v_1Pv_2x_2x_4x_3x_1v_1$ is a Hamilton cycle of $\langle Y \cup \{x_1, x_2, x_3, x_4\}\rangle_G$. By Claim 1, $\langle X\backslash\{x_1, x_2, x_3, x_4\}\rangle_G$ is hamiltonian, hence G has a 2-factor with exactly two components.

Case 2: *$\langle Y\rangle_G$ is disconnected.*

By (*ii*), $\langle Y\rangle_G$ has exactly two components D_1' and D_2' such that each of D_1' and D_2' is a clique. For each $i \in \{1, 2\}$, $\langle X \cup V(D_i')\rangle_G$ is 2-connected, thus each D_i' has a Hamilton path P^i such that the endvertices of P^i are adjacent to two distinct vertices z_1^i, z_2^i in X. If $\{z_1^1, z_2^1\} = \{z_1^2, z_2^2\}$, say, $z_1^1 = z_1^2$ and $z_2^1 = z_2^2$, then $z_1^1P^1z_2^1P^2z_1^1$ is a Hamilton cycle of $\langle Y \cup \{z_1^1, z_2^1\}\rangle_G$. Then $\langle X\backslash\{z_1^1, z_2^1\}\rangle_G$ is hamiltonian by Claim 1, implying that G has a 2-factor with exactly two components.

Hence $|\{z_1^1, z_2^1\} \cap \{z_1^2, z_2^2\}| \leq 1$. Suppose first that $|\{z_1^1, z_2^1\} \cap \{z_1^2, z_2^2\}| = 1$, say, $z_2^1 = z_1^2$. Then $z_1^1P^1z_2^1P^2z_2^2$ is a Hamilton path of $\langle Y \cup \{z_1^1, z_2^1, z_2^2\}\rangle_G$. Hence, by the definition of X, there is an edge w_1w_2 in T such that $z_1^1w_1, z_2^2w_2 \in E(G)$, and then $w_1z_1^1P^1z_2^1P^2z_2^2w_2w_1$ is a Hamilton cycle of $\langle Y \cup \{z_1^1, z_2^1, z_2^2, w_1, w_2\}\rangle_G$. By Claim 1, $\langle X\backslash\{z_1^1, z_2^1, z_2^2, w_1, w_2\}\rangle_G$ is hamiltonian, hence G has a 2-factor with exactly two components.

Thus, we have $\{z_1^1, z_2^1\} \cap \{z_1^2, z_2^2\} = \emptyset$. By the definition of X, for each $i \in \{1, 2\}$, there is an edge $y_1^iy_2^i$ in T such that $z_1^iy_1^i, y_2^iz_2^i \in E(G)$, and then $z_1^iy_1^iy_2^iz_2^iP^iz_1^i$ is a Hamilton cycle of $\langle V(D'_i) \cup \{z_1^i, y_1^i, y_2^i, z_2^i\}\rangle_G$ $(i \in \{1,2\})$. By Claim 1, $\langle X\backslash\{z_1^1, y_1^1, y_2^1, z_2^1, z_1^2, y_1^2, y_2^2, z_2^2\}\rangle_G$ is hamiltonian, hence G has a 2-factor with exactly three components.

■

3 Auxiliary results

In this section, we collect auxiliary results that will establish sufficiency parts of proofs of Theorems 2 and 3.

3.1 Sufficiency results for Theorem 2

Theorem 7. *Let* $S \in \{K_3 \cup Z_1, Z_1 \cup P_4, Z_4 \cup K_1, K_3 \cup P_4 \cup K_1, N_{1,1,1} \cup K_2\}$. *Then every 2-connected* $\{K_{1,3}, S\}$*-free graph of order at least* 2500 *has a 2-factor.*

Proof. Let, to the contrary, G be a 2-connected non-2-factorable $\{K_{1,3}, S\}$-free graph of order at least 2500 for some $S \in \{K_3 \cup Z_1, Z_1 \cup P_4, Z_4 \cup K_1, K_3 \cup P_4 \cup K_1, N_{1,1,1} \cup K_2\}$. By Theorems I and J, the class of $\{K_{1,3}, S\}$-free graphs is stable. By Theorem H, it is sufficient to consider the case that G is closed. Let H be a triangle-free graph such that $H = L^{-1}(G)$. Since $n(G) \geq 2500$, we have $e(H) \geq 2500$, and, by Theorem M, $n(H) \geq 100$. Since G is S-free, H contains no subgraph (not necessary induced) isomorphic to $L^{-1}(S)$. Recall that G is 2-connected if and only if $L^{-1}(G)$ is essentially 2-edge-connected. Since G has no 2-factor, by Lemma 5, H contains a subgraph Q isomorphic to $\Theta(k_1, k_2, k_3)$ with $k_1 \geq k_2 \geq k_3 \geq 2$ (recall that we keep the notation of its vertices as in Fig. 1(d)). Let $N_i(Q) = \{y \in V(H) \setminus V(Q) : \min\{\text{dist}_H(x, y)|\ x \in V(Q)\} = i\}$.

Claim 1. $V(H) = V(Q) \cup N_1(Q) \cup N_2(Q) \cup N_3(Q) \cup N_4(Q)$.

Proof. Suppose, to the contrary, that $N_5(Q) \neq \emptyset$. Then, by the definition of $N_i(Q)$, there is a path $P := wx_1x_2x_3x_4x_5$ in H such that $w \in V(Q)$ and $x_i \in N_i(Q)$ for $i = 1, 2, 3, 4, 5$. One can easily check that $\langle V(Q) \cup V(P)\rangle_H$ contains each of the graphs $L^{-1}(K_3 \cup Z_1), L^{-1}(Z_1 \cup P_4), L^{-1}(Z_4 \cup K_1), L^{-1}(K_3 \cup P_4 \cup K_1)$ and $L^{-1}(N_{1,1,1} \cup K_2)$ (see Fig. 2) as a subgraph, a contradiction. □

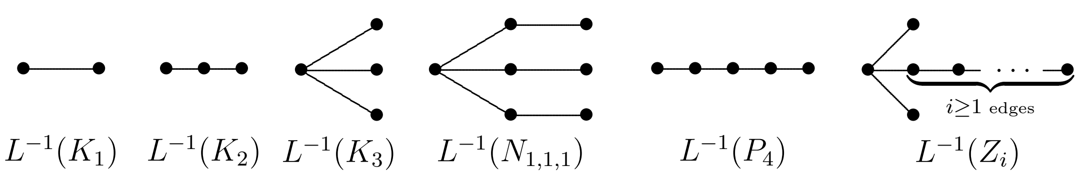

Figure 2: The preimages of the graphs from Theorem 7

Claim 2. $\sum_{j=1}^{3} k_j \leq 9$.

Proof. Let, to the contrary, $\sum_{j=1}^{3} k_j \geq 10$. Then, considering the graphs $\Theta(6,2,2)$, $\Theta(5,3,2)$, $\Theta(4,4,2)$ and $\Theta(4,3,3)$ (all Theta graphs with $\sum_{j=1}^{3} k_j = 10$ and $k_3 \geq 2$), we observe that each of them contains every graph from the set

$$\{L^{-1}(K_3 \cup Z_1), L^{-1}(Z_1 \cup P_4), L^{-1}(Z_4 \cup K_1), L^{-1}(K_3 \cup P_4 \cup K_1), L^{-1}(N_{1,1,1} \cup K_2)\}$$

as a subgraph, a contradiction. We also have the same contradiction whenever $\sum_{j=1}^{3} k_j > 10$. $\square$

By Claim 2, we have $|V(Q)| \leq 11$. We now distinguish the following two cases.

Case 1: $S \in \{K_3 \cup Z_1, Z_1 \cup P_4, Z_4 \cup K_1, K_3 \cup P_4 \cup K_1\}$.

Claim 3. *Let $x \in V(H)$. Then $|N_{H-V(Q)}(x)| \leq 1$ if $x \in N_1(Q)$, and $|N_{H-V(Q)}(x)| \leq 2$ otherwise.*

Proof. Let first $x \in V(H) \backslash N_1(Q)$, and suppose, to the contrary, that x has three neighbors x_1, x_2, x_3 outside $V(Q)$. For $x \in V(Q)$, we set $H_1 = \langle V(Q) \cup \{x_1, x_2, x_3\} \rangle_H$. For $x \in V(H) \setminus (V(Q) \cup N_1(Q))$, there is an (x, Q)-path P in H since H is connected, and we set $H_1 = \langle V(Q) \cup V(P) \cup \{x_1, x_2, x_3\} \rangle_H$. Secondly, if $x \in N_1(Q)$ has two its neighbors x_1, x_2 outside Q, we set $H_1 = \langle V(Q) \cup \{x, x_1, x_2\} \rangle_H$.
In each of the situations, the graph H_1 contains each of the graphs $L^{-1}(K_3 \cup Z_1), L^{-1}(Z_1 \cup P_4), L^{-1}(Z_4 \cup K_1)$ and $L^{-1}(K_3 \cup P_4 \cup K_1)$ as a subgraph, a contradiction. $\square$

By Claim 3, every vertex of Q has at most two neighbors outside $V(Q)$, hence $|N_1(Q)| \leq 2|V(Q)|$. Also by Claim 3, every vertex $x \in N_i(Q)$ has at most one neighbor in $N_{i+1}(Q)$, $i = 1,2,3,4$, implying that $|N_i(Q)| \leq |N_{i-1}(Q)|$ for $i = 2,3,4$ since each vertex of $N_i(Q)$ has some neighbor in $N_{i-1}(Q)$. By Claim 1, we have $n(H) \leq |V(Q)| + \sum_{i=4}^{4} |N_i(Q)| \leq 9|V(Q)|$. Then, since $|V(Q)| \leq 11$, we have $n(H) \leq 99$, contradicting the fact that $n(H) \geq 100$.

Case 2: $S = N_{1,1,1} \cup K_2$.
Since H has no subgraph isomorphic to $L^{-1}(N_{1,1,1} \cup K_2)$ and $Q - u_s$ $(s = 1,2)$ contains a subgraph isomorphic to $L^{-1}(N_{1,1,1})$, we clearly have the following two facts.

Claim 4. *For each $s \in \{1,2\}$, u_s has at most one neighbor outside $V(Q)$.*

Claim 5. *$H - V(Q)$ does not contain P_3 as a subgraph.*

If $k_1 \geq 4$, then $\langle \{u_1a_1, a_1a_2, u_1b_1, b_1b_2, u_1c_1, c_1c_2, a_{k_1-1}a_{k_1}, a_{k_1}u_2\} \rangle_H \cong L^{-1}(N_{1,1,1} \cup K_2)$, a contradiction. Hence $k_j \leq 3$ for $j = 1,2,3$. Therefore, since both $\Theta(3,3,2)$ and $\Theta(3,3,3)$ contain a subgraph isomorphic to $L^{-1}(N_{1,1,1} \cup K_2)$, we have $Q \cong \Theta(2,2,2)$ or $\Theta(3,2,2)$. We now consider the following two subcases.

<u>Subcase 2.1:</u> $Q \cong \Theta(3,2,2)$.

Since H has no subgraph isomorphic to $L^{-1}(N_{1,1,1} \cup K_2)$, it is easy to check that every vertex in $Q - a_2$ has no neighbor outside $V(Q)$. By Claim 5, $V(H) = V(Q) \cup N_1(Q) \cup N_2(Q)$. Suppose that there is a vertex $x \in N_2(Q)$. Then there is a path xya_2 in H such that $y \in N_1(Q)$. By Claim 5, x is a pendant vertex of H. Recall that H is essentially 2-edge-connected since G is 2-connected. Since every vertex of $Q - a_2$ has no neighbor outside $V(Q)$, ya_2 is a cut-edge of H and thus $H - \{ya_2\}$ has two nontrivial components, a contradiction.

Hence $N_2(Q) = \emptyset$. Then $V(H) = V(Q) \cup N_1(Q)$. Note that every vertex in $N_1(Q)$ is adjacent to a_2. Since H is triangle-free, every vertex in $N_1(Q)$ is a pendant vertex, and since $d_H(a_2) \geq 3$, $\langle E(Q - \{a_1, a_2, a_3\}), E_{a_2} \rangle_H$ is a d-system of H, a contradiction.

<u>Subcase 2.2:</u> $Q \cong \Theta(2,2,2)$.

By Claim 5, we have $V(H) = V(Q) \cup N_1(Q) \cup N_2(Q)$. If $|E(H - V(Q))| \geq 2$, then we always find a subgraph isomorphic to $L^{-1}(N_{1,1,1} \cup K_2)$ in H, a contradiction. Suppose that $|E(H - V(Q))| = 1$. Then, by Claim 5 and since H is essentially 2-edge-connected, there is an edge xy in $H - V(Q)$ such that xy has two neighbors z_1, z_2 in Q. Clearly, $\{u_1, u_2\} \cap \{z_1, z_2\} = \emptyset$ since otherwise $\langle V(Q) \cup \{x, y\} \rangle_H$ contains $L^{-1}(N_{1,1,1} \cup K_2)$ as a subgraph, a contradiction. Without loss of generality suppose that $z_1 = a_1$. For $z_2 = a_2$, we have $z_2x \notin E(H)$ since H is triangle-free, implying that $z_2y \in E(H)$. But then $\langle V(Q) \cup \{x, y\} \rangle_H$ contains $L^{-1}(N_{1,1,1} \cup K_2)$ as a subgraph, a contradiction. For $z_2 \in \{b_1, c_1\}$, say, $z_2 = b_1$, we set $C := u_1a_1xyb_1b_2u_2c_2c_1u_1$ when $yb_1 \in E(H)$ (or $C := u_1a_1xb_1b_2u_2c_2c_1u_1$ otherwise). Clearly C is a cycle in H. If a_2 has no neighbors outside Q, C is dominating in H, implying that H has a d-system, a contradiction. If a_2 has some neighbors outside Q, then C together with E_{a_2} is a d-system in H, a contradiction again. For $z_2 \in \{b_2, c_2\}$, say, $z_2 = b_2$, $C := u_1c_1c_2u_2a_2a_1xyb_2b_1u_1$ when $z_2y \in E(G)$ (or $C := u_1c_1c_2u_2a_2a_1xb_2b_1u_1$ otherwise) is a dominating cycle in H, implying that G is hamiltonian, a contradiction.

Hence $V(H) = V(Q) \cup N_1(Q)$ and $N_1(Q)$ is an independent set of H. Since $|V(Q)| = 8$ and $n(H) = |V(Q)| + |N_1(Q)| \geq 100$, we have $|N_1(Q)| \geq 100 - 8 = 92$. Since every vertex in $N_1(Q)$ has a neighbor in Q and $|V(Q)| = 8$, there is a vertex v of Q such that v has at least 12 neighbors in $N_1(Q)$. By Claim 4, $v \notin \{u_1, u_2\}$, hence $v \in \{a_1, a_2, b_1, b_2, c_1, c_2\}$. Without loss of generality, we may assume that $v = a_1$. Denote three neighbors v_1, v_2, v_3 of a_1 in $N_1(Q)$. Then a_2 has no neighbor in $N_1(Q)$, since otherwise, for some $w \in N_{N_1(Q)}(a_2)$, $\langle E(Q) \cup \{a_1v_1, a_1v_2, a_2w\} \rangle_H$ contains a subgraph isomorphic to $L^{-1}(N_{1,1,1} \cup K_2)$. Then $\langle E(Q - \{a_1, a_2\}), E_{a_1} \rangle_H$ is a d-system of H, a contradiction.

■

Theorem 8. *Every 2-connected $\{K_{1,k}, 3K_1 \cup K_2\}$-free graph, $k \geq 2$, of order at least $R(3k + 26, k+2)$ has a 2-factor.*

Proof. We claim that G is $(k+2)K_1$-free. Let, to the contrary, $S = \{v_1, v_2, \cdots, v_{k+2}\}$ be an independent set in G. Since G is connected, there is a vertex u in $G - S$ such that $uv_1 \in E(G)$. Since G is $K_{1,k}$-free and S is an independent set, u has at most $k-1$ neighbors in S. This implies that there exists a triple of vertices, say v_k, v_{k+1}, v_{k+2}, in S such that $uv_i \notin E(G)$, and then $\{v_k, v_{k+1}, v_{k+2}, v_1, u\}$ induces a $3K_1 \cup K_2$, a contradiction. Therefore, since $n(G) \geq R(3k+26, k+2)$, G contains a clique T of order $3k+26$. Set

$$X = \{x \in V(G) : d_T(x) \geq k+9\} \text{ and } Y = V(G) \setminus X.$$

We now claim that $\alpha(\langle Y \rangle_G) \leq 2$. Let, to the contrary, $\{y_1, y_2, y_3\}$ be an independent set in $\langle Y \rangle_G$. Then, by the definition of Y, y_i has at most $k+8$ neighbors in T, $1 \leq i \leq 3$. Since $|V(T)| \geq 3k+26$, there is an edge x_1x_2 in T such that none of y_i ($i = 1, 2, 3$) is adjacent to any of x_1, x_2. However, $\{y_1, y_2, y_3, x_1, x_2\}$ induces a $3K_1 \cup K_2$, a contradiction. Thus, G satisfies the assumptions of Lemma 6, and hence it has a 2-factor. ■

Theorem 9. *Every 2-connected $\{kK_1, 3K_1 \cup K_l\}$-free graph, $k \geq 4$, $l \geq 2$, of order at least $R(3k+l+18, k)$ has a 2-factor.*

Proof. Since G is kK_1-free and $n(G) \geq R(3k+l+18, k)$, G contains a clique T of order $3k+l+18$. Set

$$X = \{x \in V(G) : d_T(x) \geq k+7\} \text{ and } Y = V(G) \setminus X.$$

We now claim that $\alpha(\langle Y \rangle_G) \leq 2$. Let, to the contrary, $\{y_1, y_2, y_3\}$ be an independent set in $\langle Y \rangle_G$. By the definition of Y, y_i has at most $k+6$ neighbors in T, $1 \leq i \leq 3$. Since $|V(T)| \geq 3k+l+18$, there is a subgraph T' of T such that $|V(T')| \geq l$ and no vertex in T' is adjacent to any of $\{y_1, y_2, y_3\}$. Then $\{y_1, y_2, y_3\} \cup V(T')$ induces a $3K_1 \cup K_l$, a contradiction. Thus, G satisfies the assumptions of Lemma 6, and hence G has a 2-factor. ■

Theorem 10. *Let G be a 2-connected $\{K_{1,4}, P_3 \cup 2K_1\}$-free graph of order at least $R(113, 5)$. Then G has a 2-factor.*

Proof. We start the proof with the following statement.

Claim 1. *G is $5K_1$-free.*

Proof. Let, to the contrary, $v_1, v_2, \ldots, v_5$ be an induced $5K_1$ in G. Since G is connected, there is a path P between v_1 and some of the vertices v_2, v_3, v_4, v_5. Choose P shortest possible and choose the notation of the vertices such that P is a (v_1, v_2)-path. Hence none of v_3, v_4, v_5 belongs to P. Then $|V(P)| \leq 7$, for otherwise P contains an induced $P_3 \cup 2K_1$. On the other hand, $|V(P)| \geq 3$, for otherwise $v_1v_2 \in E(G)$. Hence $3 \leq |V(P)| \leq 7$. Let x denote the neighbor of v_2 on P. By the choice of P, none of v_3, v_4, v_5 is adjacent to any internal vertex of P distinct from x (if any), and since G is $K_{1,4}$-free, x is adjacent to at most one of v_3, v_4, v_5, say, to v_5. Then v_3, v_4 and the subpath of P of length 2 with one endvertex v_1 induce a $P_3 \cup 2K_1$, a contradiction. $\square$

Since G is $5K_1$-free and $n \geq R(113, 5)$, G contains a clique T of order 113. Set $X = \{x \in V(G), d_T(x) \geq 17\}$ and $Y = V(G) \setminus X$. Clearly $V(T) \subseteq X$. For $Y = \emptyset$, we know that G is hamiltonian by Theorem N. Hence assume that $Y \neq \emptyset$. If $\alpha(\langle Y \rangle_G) \leq 2$, then G has a 2-factor by Lemma 6 since each $5K_1$-free graph is also $10K_1$-free. Thus, in the rest of the proof, we assume that $\alpha(\langle Y \rangle_G) \geq 3$.

Claim 2. $\alpha(\langle Y \rangle_G) = 3$.

Proof. Let, to the contrary, $\alpha(\langle Y \rangle_G) \geq 4$, and let $I = \{y_1, y_2, y_3, y_4\}$ be an independent set in Y. Since each of y_i ($i = 1, 2, 3, 4$) has at most 16 neighbors in T (by the definition of Y) and T has 113 vertices, T contains a vertex t such that $ty_i \notin E(G)$ for every $i = 1, 2, 3, 4$. This implies that y_1, y_2, y_3, y_4, t induce a $5K_1$, contradicting Claim 1. $\square$

Claim 3. *$\langle Y \rangle_G$ has no induced subgraph isomorphic to $P_3 \cup K_1$.*

Proof. Let, to the contrary, $y_1y_2y_3, y_4$ be an induced $P_3 \cup K_1$ in $\langle Y \rangle_G$. Since each of y_i ($i = 1, 2, 3, 4$) has at most 16 neighbors in T (by the definition of Y) and T has 113 vertices, there is a vertex t in T such that $ty_i \notin E(G)$ for every $i = 1, 2, 3, 4$. Then $\{y_1, y_2, y_3, y_4, t\}$ induces a $P_3 \cup 2K_1$, a contradiction. $\square$

Claim 4. *For any $X' \subset X$ with $|X'| \leq 12$, $\langle X \setminus X' \rangle_G$ is Hamilton-connected.*

Proof. Let $X' \subset X$ with $|X'| \leq 12$ and let $G' = \langle X \setminus X' \rangle_G$. For G' we have $\kappa(G') \geq \kappa(G) - 12 \geq 17 - 12 = 5$, and $\alpha(G') \leq \alpha(G) \leq 4$ by Claim 1. Thus G' is Hamilton-connected by Theorem N. $\square$

Now we consider the following two cases.

Case 1: *$\langle Y \rangle_G$ is disconnected.*

By Claim 2, $\mathrm{nc}(\langle Y \rangle_G) \leq 3$. First assume that $\langle Y \rangle_G$ consists of two components, denoted D_1 and D_2. Then one of D_1, D_2, say, D_1, is a clique, and D_2 is of diameter 2 or 3, since $\alpha(\langle Y \rangle_G) = 3$. Let $y_1 \in V(D_1)$ and let P be an induced path in D_2 of length 2. This yields an induced $P_3 \cup K_1$ in $\langle Y \rangle_G$, contradicting Claim 3.

Hence $\langle Y \rangle_G$ consists of three components, denoted D_1, D_2, D_3. By Claim 2, each D_i ($i = 1, 2, 3$) is a clique. Since G is 2-connected and since D_1, D_2, D_3 are cliques, for each $i = 1, 2, 3$, there are two distinct vertices x_i^1, x_i^2 in X (possibly $x_{i_1}^{j_1} = x_{i_2}^{j_2}$ for some $i_1, i_2 \in \{1, 2, 3\}$, $j_1, j_2 \in \{1, 2\}$, $i_1 \neq i_2$) such that $\langle \{x_i^1, x_i^2\} \cup V(D_i) \rangle_G$ has a Hamilton (x_i^1, x_i^2)-path Q_i. Let $M = \{x_1^1, x_1^2, x_2^1, x_2^2, x_3^1, x_3^2\}$. Then $2 \leq |M| \leq 6$. Choose the vertices x_i^1, x_i^2 ($i = 1, 2, 3$) such that $|M|$ is maximal.

Let $Q = Q_1 \cup Q_2 \cup Q_3$. Suppose that, say, $x_1^1 = x_2^1 = x_3^1$. Let $(x_1^1)^i$ denote the successor of x_1^1 on Q_i, $i = 1, 2, 3$, and let $x \in T \setminus M$ such that $xx_1^1 \in E(G)$. If x is adjacent to none of $(x_1^1)^i$, $i = 1, 2, 3$, then $\langle \{x_1^1, (x_1^1)^1, (x_1^1)^2, (x_1^1)^3, x\} \rangle_G$ is an induced $K_{1,4}$, a contradiction. Hence x is adjacent to some of $(x_1^1)^i$, say, to $(x_1^1)^1$. Then considering a Hamilton $((x, x_1^2))$-path Q_1' in $\langle \{x, x_1^2\} \cup V(D_1) \rangle_G$ instead of Q_1 contradicts the maximality of $|M|$.

Hence $x_a^b = x_c^d = x_e^f$ for no triple of vertices from M ($a, c, e \in \{1, 2, 3\}$, $b, d, f \in \{1, 2\}$). Therefore Q consists of at most three components and each of them is a path or a cycle. Similarly as in the proof of Lemma 6, since T is a clique of order 113, there is an edge $w_i^1 w_i^2$ in T such that $w_i^j x_i^j \in E(G)$, $i = 1, 2, 3$ and $j = 1, 2$. Set $N = \{w_i^j\}$. Then $\langle V(Q) \cup N \rangle_G$ has a 2-factor. Since $|M \cup N| \leq 12$, $\langle X \setminus (M \cup N) \rangle_G$ is hamiltonian by Claim 4, implying that G has a 2-factor.

Case 2: *$\langle Y \rangle_G$ is connected.*

Let V denote a minimal vertex cut in $\langle Y \rangle_G$. The following fact is obvious by Claims 2 and 3.

Claim 5. *The subgraph $\langle Y \rangle_G - V$ consists of at most three components, and these components are all cliques.*

If $\kappa(\langle Y \rangle_G) \geq 3$, then $\langle Y \rangle_G$ and $\langle X \rangle_G$ are both hamiltonian by Theorem N, implying that G has a 2-factor. Hence we assume that $1 \leq \kappa(\langle Y \rangle_G) \leq 2$. We now consider the following two subcases.

Subcase 2.1: $\kappa(\langle Y \rangle_G) = 2$.

Let $V = \{v_1, v_2\}$. By Claim 5, $\langle Y \rangle_G - \{v_1, v_2\}$ consists of at most three components D_1, D_2, D_3 (D_3 may be empty), and each D_i ($i = 1, 2, 3$) is a clique. If D_3 is empty, then, since $\langle Y \rangle_G$ is 2-connected, $\langle Y \rangle_G$ is hamiltonian, implying that G has a 2-factor. Hence we assume that D_3 is nonempty. Then we have the following fact.

Claim 6. *Each of v_1, v_2 is adjacent to every vertex in $D_1 \cup D_2 \cup D_3$.*

Proof. Let, to the contrary, $y_i v_j \notin E(G)$ for some $y_i \in V(D_i)$, $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$, say, $i = j = 1$. Then there are $y_2 \in V(D_2)$ and $y_3 \in V(D_3)$ such that $y_3 v_1 y_2$ is an induced path in G since $\{v_1, v_2\}$ is a minimal vertex cut of $\langle Y \rangle_G$. However, $\{y_3, v_1, y_2, y_1\}$ induces a $P_3 \cup K_1$, contradicting Claim 3. □

Now, since G is 2-connected, there are two disjoint edges $x_i y_i$ ($i = 1, 2$) between some vertices $x_i \in X$ and $y_i \in Y$. Choose edges $x_1 y_1$, $x_2 y_2$ such that $|\{y_1, y_2\} \cap \{v_1, v_2\}|$ is minimal. The following possibilities can occur.

(i) *Both y_1 and y_2 belong to the same component of $\langle Y \rangle_G - \{v_1, v_2\}$.*
Let D_1 be such a component. Then $\langle V(D_1) \cup \{x_1, x_2\} \rangle_G$ has a Hamilton (x_1, x_2)-path, and, by Claim 6, $\langle V(D_2) \cup V(D_3) \cup \{v_1, v_2\} \rangle_G$ is hamiltonian, implying that G has a 2-factor since $\langle X \rangle_G$ is Hamilton-connected by Claim 4.

(ii) *The vertices y_1, y_2 belong to distinct components of $\langle Y\rangle_G - \{v_1, v_2\}$.*
Without loss of generality suppose that $y_1 \in V(D_1)$ and $y_2 \in V(D_3)$. Then, by Claim 6, there is a Hamilton (x_1, x_2)-path in $\langle Y \cup \{x_1, x_2\}\rangle_G$, implying that G has a 2-factor since $\langle X\rangle_G$ is Hamilton-connected by Claim 4.

(iii) $\{y_1, y_2\} \cap \{v_1, v_2\} \neq \emptyset$.
Without loss of generality suppose that $v_1 = y_1$. Then, since G is $K_{1,4}$-free, x_1 is adjacent to every vertex of some D_i, say, of D_1. Thus $y_2 \notin V(D_2) \cup V(D_3) \cup \{v_2\}$, for otherwise, considering any vertex in D_1 instead of y_1 contradicts the choice of x_1y_1, x_2y_2. This implies that $y_2 \in V(D_1)$. Take two vertices $t_1, t_2 \in V(T)$ such that $x_it_i \in E(G)$. Then $\langle V(D_1) \cup \{x_1, x_2, t_1, t_2\}\rangle_G$ is hamiltonian. By Claim 4, $\langle X \setminus \{x_1, x_2, t_1, t_2\}\rangle_G$ is hamiltonian. By Claim 6, $\langle V(D_2) \cup V(D_3) \cup \{v_1, v_2\}\rangle_G$ is hamiltonian. Then G has a 2-factor with exactly three components.

Subcase 2.2: $\kappa(\langle Y\rangle_G) = 1$.

Let $V = \{v\}$. By Claim 5, $\langle Y\rangle_G - v$ consists of at most three components D_1, D_2, D_3 (D_3 may be empty) and each D_i $(i = 1, 2, 3)$ is a clique. Suppose that D_3 is empty. Then, since G is 2-connected, there is a pair of vertex-disjoint edges x_1y_1 and x_2y_2 such that $x_1, x_2 \in X$ and $y_i \in V(D_i)$ $(i = 1, 2)$. Clearly, $\langle Y\rangle_G$ has a Hamilton (y_1, y_2)-path, and since $\langle X\rangle_G$ is Hamilton-connected by Claim 4, there is a Hamilton (x_1, x_2)-path in $\langle X\rangle_G$. Thus, G is hamiltonian.

Hence suppose that D_3 is nonempty. Then the following fact is obvious by Claim 3.

Claim 7. *The vertex v is adjacent to every vertex in Y.*

Suppose that some of the components D_i, say, D_3, contains more than two vertices. Clearly D_3 is hamiltonian. Since G is 2-connected, there is an edge x_iy_i such that $x_i \in X$ and $y_i \in V(D_i)$ for $i = 1, 2$. If $x_1 = x_2$, then, by Claim 7, $\langle V(D_1) \cup V(D_2) \cup \{x_1\}\rangle_G$ is hamiltonian. By Claim 4, $\langle X \setminus \{x_1\}\rangle_G$ is hamiltonian, and then G has a 2-factor. If $x_1 \neq x_2$, then, by Claim 7, $\langle V(D_1) \cup V(D_2) \cup \{x_1, x_2\}\rangle_G$ has a Hamilton (x_1, x_2)-path. Since $\langle X\rangle_G$ is Hamilton-connected by Claim 4, there is a Hamilton (x_1, x_2)-path in $\langle X\rangle_G$, hence G has a 2-factor with exactly two components.

Hence suppose that $|V(D_i)| \leq 2$ for each $i = 1, 2, 3$. Then $|Y| \leq 7$. By the definition of Y, every vertex in Y has at most 16 neighbors in T. Since T has 113 vertices, $V(T)\setminus N_T(Y) \neq \emptyset$. Let P be a shortest path between some vertex of $Y \setminus \{v\}$ and some vertex y_P of $V(T) \setminus N_T(Y)$ (possibly $N_T(Y) = \emptyset$). We may assume that $y_P \in D_1$. If $N_T(Y) = \emptyset$, then y_P has a neighbor x_P in X, and, considering any neighbor t_P of x_P in T, we get $P = t_Px_Py_P$. On the other hand, if $N_T(Y) \neq \emptyset$, then y_P has a neighbor x_P in T, which is adjacent to each vertex of $V(T) \setminus N_T(Y)$, thus also to t_P. Obviously, P has length 2.

Claim 8. *There is $j \in \{2, 3\}$ such that $V(D_1) \cup V(D_j) \subset N_G(x_p)$ and $V(D_{5-j}) \cap N_G(x_p) = \emptyset$.*

Proof. Let, say, $j = 2$. If in each of $D_{j'}$ $(j' = 2, 3)$ there is a vertex $y_{j'}$ such that $x_p \notin N_G(y_{j'})$, then G contains an induced $P_3 \cup 2K_1$, a contradiction. Hence x_p is adjacent to every vertex of one of $D_{j'}$, say, of D_2. If some vertex of D_3 is adjacent to x_P, then x_P is the center of an induced $K_{1,4}$, a contradiction. Thus there is no edge between x_P and $V(D_3)$. By a symmetric argument, each vertex of D_1 is adjacent to x_P. □

Since G is 2-connected, there is an induced path $Q = y_Q x_Q t_Q$ in G such that $y_Q \in V(D_3), x_Q \in X$ and $t_Q \in V(T)$. By Claim 8, $x_P \neq x_Q$. Since G is $P_3 \cup 2K_1$-free, x_Q (or t_Q) is adjacent to some vertex of $D_1 \cup D_2$. Then, by Claims 7 and 8, $\langle Y \cup \{x_P, x_Q\}\rangle_G$ (or $\langle Y \cup \{x_P, x_Q, t_Q\}\rangle_G$) is hamiltonian, and, by Claim 4, $\langle X \setminus \{x_P, x_Q\}\rangle_G$ (or $\langle X \setminus \{x_P, x_Q, t_Q\}\rangle_G$) is hamiltonian, implying that G has a 2-factor with exactly two components. ■

3.2 Sufficiency results for Theorem 3

Theorem 11. *Every connected $\{K_{1,3}, S\}$-free graph of order at least 2500 and minimum degree at least two has a 2-factor for any $S \in \{P_3 \cup K_2, Z_1 \cup K_2, K_1 \cup K_2 \cup K_3\}$.*

Proof. If G is 2-connected, then G has a 2-factor by Theorem 7. Hence we only consider the case that $\kappa(G) = 1$. Let v be a cut-vertex of G. Then $G - v$ has exactly two components since G is claw-free. If $S = P_3 \cup K_2$, then each component of $G - v$ is a clique since $n(G) \geq 6$ and G is $P_3 \cup K_2$-free, implying that G has a 2-factor. It remains to consider the following two cases.

Case 1: $S = Z_1 \cup K_2$.

Suppose first that G has a cut-edge x_1x_2. Then $G - x_1x_2$ has two components D_1, D_2 with $x_i \in V(D_i)$, $i = 1, 2$. Since $\delta(G) \geq 2$ and G is claw-free, each of D_1, D_2 has a triangle. If, say, $d_{D_1}(x_1) = 1$, we choose a shortest (x_1, y)-path P such that y is in a triangle, say, T. Then $V(T) \cup V(P)$ contains an induced Z_1 in D_1. Together with an edge in $D_2 - x_2$ we have an induced $Z_1 \cup K_2$, a contradiction.

Hence for each $i \in \{1, 2\}$, x_i has at least two neighbors in D_i, implying that $\delta(D_i) \geq 2$. Since $\delta(G) \geq 2$, we have $|V(D_i)| \geq 3$ for $i = 1, 2$. Therefore, since G is $Z_1 \cup K_2$-free, each D_i $(i = 1, 2)$ is $\{K_{1,3}, Z_1\}$-free. By Corollary 4, both D_1 and D_2 are hamiltonian, hence G has a 2-factor.

Now suppose that G is 2-edge-connected. Since G is claw-free and $\delta(G) \geq 2$, each block of G contains a triangle. If G has more than two blocks, then using two appropriate blocks for Z_1 and one for K_2 we get an induced $Z_1 \cup K_2$, a contradiction. Thus G has two blocks B_1, B_2 and a cut-vertex v. Then each B_i $(i = 1, 2)$ is $\{K_{1,3}, Z_1\}$-free, and, by Corollary 4, both B_1 and B_2 are hamiltonian, implying that G has a 2-factor since $n(G) \geq 2500$.

Case 2: $S = K_1 \cup K_2 \cup K_3$.

Suppose first that G has a cut-edge x_1x_2. Then $G - x_1x_2$ has two components D_1, D_2 with $x_i \in V(D_i)$, $i = 1, 2$. Assume that $d_{D_i}(x_i) = 1$ for some $i \in \{1, 2\}$, say, for $i = 1$. Let y denote the neighbor of x_1 in D_1. Since $\delta(G) \geq 2$ and G is claw-free, each of D_1, D_2 has a triangle. Then each of D_1 and $\langle \{x_1\} \cup V(D_2) \rangle_G$ contains an induced $K_1 \cup K_2$.

We now show that $|N_{D_1 - x_1}(y)| = |N_{D_2}(x_2)| = 2$. If, say, x_2 has at least three neighbors in D_2, then $\langle N_{D_2}(x_2) \rangle_G$ contains a triangle since G is claw-free, and together with an induced $K_1 \cup K_2$ in D_1 we have an induced $K_1 \cup K_2 \cup K_3$ in G, a contradiction. Hence $|N_{D_2}(x_2)| \leq 2$, and, symmetrically, $|N_{D_1 - x_1}(y)| \leq 2$.

Now, if, say, $N_{D_2}(x_2) = \{x\}$, then x_2x is a cut-edge of G. Since $\delta(G) \geq 2$, there is a K_3 in $D_2 - x_2$, and together with an induced $K_1 \cup K_2$ in D_1 we have an induced $K_1 \cup K_2 \cup K_3$ in G, a contradiction. Hence $|N_{D_2}(x_2)| = 2$, and, symmetrically, $|N_{D_1 - x_1}(y)| = 2$.

Let $N_{D_1 - x_1}(y) = \{y_1, y_2\}$ and $N_{D_2}(x_2) = \{z_1, z_2\}$. Then $y_1y_2, z_1z_2 \in E(G)$ since G is claw-free. Since $n(G) \geq 8$, there is a vertex $w \in V(G) \setminus \{x_1, x_2, y, y_1, y_2, z_1, z_2\}$ adjacent to some of $\{y_1, y_2, z_1, z_2\}$, say z_1. Then $wz_2 \notin E(G)$, for otherwise $\{x_1, y_1, y_2, w, z_1, z_2\}$ induces a $K_1 \cup K_2 \cup K_3$, a contradiction. Therefore, since $\delta(G) \geq 2$, w has a neighbor w' in $D_2 - \{x_2, z_1, z_2\}$, and then $\{y, y_1, y_2, x_2, w, w'\}$ induces a $K_1 \cup K_2 \cup K_3$, a contradiction.

Hence for each $i = 1, 2$, x_i has at least two neighbors in D_i, thus $\delta(D_i) \geq 2$. Recall that each D_i $(i = 1, 2)$ contains a triangle. Since G is $K_1 \cup K_2 \cup K_3$-free, $D_i - x_i$ is $K_1 \cup K_2$-free $(i = 1, 2)$, implying that D_i is Z_2-free. Then, by Theorem P and since $D_i - x_i$ is $K_1 \cup K_2$-free, D_i is hamiltonian, implying that G has a 2-factor.

Now suppose that G is 2-edge-connected. Since G is claw-free, every cut-vertex of G belongs to two blocks of G. Note that each block of G contains a triangle. Since G is $K_1 \cup K_2 \cup K_3$-free, G has at most three blocks. If G has exactly three blocks B_1, B_2 and B_3, then since G is $K_1 \cup K_2 \cup K_3$-free, each B_i $(1 \leq i \leq 3)$ is clique. Since $n(G) \geq 9$, it is easy to see that G contains an induced $K_1 \cup K_2 \cup K_3$, a contradiction.

Hence G has exactly two blocks B_1, B_2 and a cut-vertex v. If each of B_1 and B_2 is hamiltonian, then G contains a 2-factor since $n(G) \geq 2500$. Hence at least one of B_1, B_2, say, B_1, is not hamiltonian. By Theorem A, B_1 contains an induced P_6, let P denote such a path. Since G is claw-free, $N_{B_1}(v)$ induces a clique in B_1, implying that $|N_{B_1}(v) \cap V(P)| \leq 2$. Then $B_1 - (\{v\} \cup N_{B_1}(v))$ contains an induced $K_1 \cup K_2$, implying that G contains an induced $K_1 \cup K_2 \cup K_3$ since B_2 has a triangle, a contradiction.

■

Theorem 12. *Every connected $\{K_{1,k}, 2K_1 \cup K_2\}$-free graph, $k \geq 4$, of order at least $R(3k + 26, k + 2)$ and minimum degree at least two has a 2-factor.*

Proof. If G is 2-connected, then, since every $2K_1 \cup K_2$-free graph is also $3K_1 \cup K_2$-free, G has a 2-factor by Theorem 8. Thus we only consider the case $\kappa(G) = 1$. Let v be a cut-vertex of G. Then $G-v$ has at most $k-1$ components. Since $\delta(G) \geq 2$, every component of $G-v$ has at least two vertices. Therefore, since G is $2K_1 \cup K_2$-free, $G-v$ has exactly two components D_1, D_2 and each D_i is a clique. Since $n(G)$ is large and G is $2K_1 \cup K_2$-free, v has at least two neighbors in some D_i, and then it is easy to see that G has a 2-factor. ■

Theorem 13. *Every connected $\{kK_1, 2K_1 \cup K_l\}$-free graph, $k \geq 4$, $l \geq 2$, of order at least $R(2k+l+4, k)$ and minimum degree at least two has a 2-factor.*

Proof. Since G is kK_1-free and $n(G) \geq R(2k+l+4, k)$, G contains a clique T of order $2k+l+4$. Set

$$X = \{x \in V(G) : d_T(x) \geq k+3\} \text{ and } Y = V(G) \backslash X.$$

Claim 1. *For any set $X' \subset X$ with $|X'| \leq 4$, $\langle X \setminus X' \rangle_G$ is hamiltonian.*

Proof. We have $\alpha(\langle X \setminus X' \rangle_G) \leq \alpha(\langle X \rangle_G) \leq k-1$ and $\kappa(\langle X \setminus X' \rangle_G) \geq \kappa(\langle X \rangle_G) - 4 \geq k+3-4 = k-1$. By Theorem N, $\langle X \setminus X' \rangle_G$ is hamiltonian. □

We now claim that $\langle Y \rangle_G$ is a clique. Let, to the contrary, u_1, u_2 be a pair of nonadjacent vertices in Y. By the definition of Y, each u_i $(i = 1, 2)$ has at most $k+2$ neighbors in T. Since $|V(T)| \geq 2k+l+4$, there is a subgraph T' of T such that $|V(T')| \geq l$ and each vertex in T' is nonadjacent to any of $\{u_1, u_2\}$. Then $\{u_1, u_2\} \cup V(T')$ induces a $2K_1 \cup K_l$, a contradiction. If Y has at least three vertices, then clearly $\langle Y \rangle_G$ is hamiltonian, implying that G has a 2-factor by Claim 1. Hence we assume that Y has at most two vertices y_1, y_2 (possibly $y_1 = y_2$). Since $\delta(G) \geq 2$, each y_i $(i = 1, 2)$ has a neighbor x_i in X (possibly $x_1 = x_2$), or, in the case when $y_1 = y_2$, y_1 has at least two distinct neighbors x_1, x_2 in X. Let z_i be a neighbor of x_i in T for $i = 1, 2$. Let $Y' = Y \cup \{x_1\}$ when $x_1 = x_2$, or $Y' = Y \cup \{x_1, x_2, z_1, z_2\}$ otherwise. Then $\langle Y' \rangle_G$ is hamiltonian as well as $\langle V(G) \setminus Y' \rangle_G$ is hamiltonian by Claim 1, implying that G has a 2-factor. ■

Theorem 14. *Every connected $\{4K_1, K_1 \cup K_2 \cup K_l\}$-free graph, $l \geq 2$, of order at least $\max\{R(l+4, 4), R(31, 4)\}$ and minimum degree at least two has a 2-factor.*

Proof. Since G is $4K_1$-free and $n(G) \geq R(l+4, 4)$, G contains a clique of order $l+4$. If G is 2-connected, then G has a 2-factor by Theorem 1. Hence we only consider the case that $\kappa(G) = 1$.

Suppose first that G has a cut-edge x_1x_2. Then $G - x_1x_2$ has two components D_1, D_2, thus one of D_1, D_2, say, D_1, contains a clique of order $l+4$. Since $\delta(G) \geq 2$, we have $|V(D_2)| \geq 3$. Since $D_1 - x_1$ has a K_{l+3} and G is $K_1 \cup K_2 \cup K_l$-free, D_2 is $K_1 \cup K_2$-free.

If x_1 has only one neighbor x in D_1, then $D_1 - \{x_1, x\}$ contains a K_{l+2}, implying that D_1 contains an induced $K_1 \cup K_{l+2}$. But then G contains an induced $K_1 \cup K_2 \cup K_{l+2}$ since $|V(D_2)| \geq 3$, a contradiction. Similarly, if x_2 has only one neighbor y in D_2, then $D_2 - y$ contains an edge since $\delta(G) \geq 2$, implying that D_2 contains an induced $K_1 \cup K_2$, a contradiction. Thus $\delta(D_i) \geq 2$, $i = 1, 2$.

Since $\alpha(G) \leq 3$, we have $\alpha(D_i) \leq 2$, $i = 1, 2$. Since $|V(D_1)| \geq l + 4 \geq 6$, D_1 has a 2-factor by Theorem F. Since D_2 is $\{K_{1,3}, K_1 \cup K_2\}$-free, D_2 is hamiltonian by Corollary 4. Hence G has a 2-factor.

Now suppose that G is 2-edge-connected. Since $\alpha(G) \leq 3$ and $\kappa(G) = 1$, G has two or three blocks, each of order at least 3, and 1 or 2 cut-vertices. If G has three blocks, then, since $n(G) \geq R(l + 4, 4)$, one of the blocks contains a K_{l+4}, and we easily find an induced $K_1 \cup K_2 \cup K_l$ in G. Thus, we suppose that G has 2 blocks B_1, B_2 and one cut-vertex v. Since $\alpha(G) \leq 3$, one of B_1, B_2, say, B_2, is a clique and B_1 is hamiltonian by Theorem N, and then G has a 2-factor with 2 components, unless B_2 is a triangle. Thus, let $V(B_2) = \{v, v_1, v_2\}$. Since $n(G) \geq R(l + 4, 4)$, B_1 contains a clique K of order at least $l + 4$. If there is a vertex $x \in V(B_1) \setminus (V(K) \cup \{v\})$ having at most 3 neighbors in K, then $\langle\{x\}\rangle_G \cup \langle\{v_1, v_2\}\rangle_G \cup \langle V(K) \setminus (N_K(x) \cup \{v\})\rangle_G$ gives an induced $K_1 \cup K_2 \cup K_l$ in G. Thus, every vertex in $V(B_1) \setminus (V(K) \cup \{v\})$ has at least 4 neighbors in K. Then $B_1 - v$ is 2-connected, hence hamiltonian by Theorem N, and a Hamilton cycle in $B_1 - v$ and the triangle vv_1v_2 yield a 2-factor in G. ■

4 Proofs of the main results

Given any integer n_0, we consider the nine 2-connected non-2-factorable graphs G_i of order at least n_0 shown in Fig. 3.

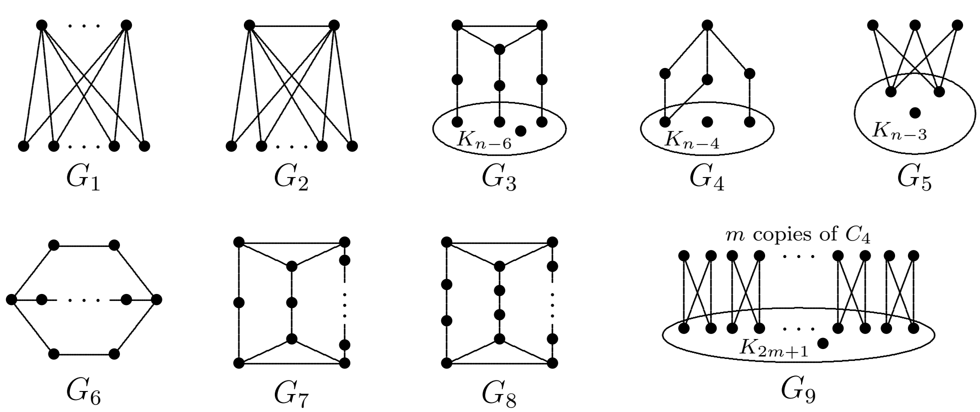

Figure 3: 2-connected non-2-factorable graphs of arbitrarily large order

Proof of Theorem 1.

Necessity. For each $i \in \{1, 2, 3, 4\}$, G_i is non-2-factorable of order at least $R(31, 4)$ and hence it contains F as an induced subgraph. If F is connected, then, since the largest common connected induced subgraph of G_1, G_2 and G_3 is P_3, F is an induced subgraph of P_3. If F is disconnected, then, since every disconnected induced subgraph of G_1 is edgeless and the independence number of G_4 is 4, F is an induced subgraph of $4K_1$.

Sufficiency. Let G be a 2-connected graph of order at least $R(31,4)$. If G is P_3-free, then G is complete and hence hamiltonian. Hence we assume that G is $4K_1$-free. Since $n(G) \geq R(31,4)$, G contains a clique T of order 31. Let $X = \{x \in V(G) : d_T(x) \geq 11\}$ and $Y = V(G) \backslash X$. Clearly $T \subset X$. We claim that $\alpha(\langle Y \rangle_G) \leq 2$. Let, to the contrary, $\{y_1, y_2, y_3\}$ be an independent set in Y. By the definition of Y, each y_i ($1 \leq i \leq 3$) has at most 10 neighbors in T. Since the order of T is 31, there is a vertex x in T such that x is nonadjacent to any of $\{y_1, y_2, y_3\}$. Then $\{x, y_1, y_2, y_3\}$ is an independent set of G, contradicting the fact that G is $4K_1$-free. Thus G satisfies the conditions of Lemma 6, implying that G has a 2-factor. ■

Proof of Theorem 2.

Combining Theorems C and E, sufficiency follows from Theorems 7, 8, 9, 10 and O. Hence it remains to show necessity.

Let R, S be a pair of graphs of order at least three other than $P_3, 3K_1$ and $4K_1$. Consider the graphs $G_1, \ldots, G_9$ shown in Fig. 3. For each $1 \leq i \leq 9$, G_i is non-2-factorable of arbitrarily large order and hence it contains at least one of R, S as an induced subgraph.

We now show that either R or S is edgeless or a star. Suppose, to the contrary, that neither R nor S is edgeless or a star, and recall that each of R and S is not an induced subgraph of P_3 and $4K_1$. If, say, $|V(R)| \leq 3$, then R is K_3 or $K_1 \cup K_2$, and if $|V(R)| \geq 4$, then R contains an induced $K_1 \cup K_2$ when R is disconnected or a tree, or any induced cycle in R contains an induced K_3, C_4 or a $K_1 \cup K_2$. Thus, in any case, the graph R (and symmetrically also S) contains some of $K_3, C_4, K_1 \cup K_2$ as an induced subgraph. We may assume, without loss of generality, that R is an induced subgraph of G_1. Since G_1 is $\{K_3, K_1 \cup K_2\}$-free, R contains C_4 as an induced subgraph. Since G_2 is C_4-free, G_2 contains S as an induced subgraph, and since G_2 is $\{C_4, K_1 \cup K_2\}$-free, S contains K_3 as an induced subgraph. But then G_6 is $\{K_3, C_4\}$-free, implying that G_6 is $\{R, S\}$-free and hence it has a 2-factor, a contradiction.

In the rest of the proof we assume (up to a symmetry) that R is edgeless or a star. Now we consider the following four cases.

Case 1: $R = K_{1,3}$.

For each $i \in \{3, 7, 8\}$, G_i is $K_{1,3}$-free and then it contains S as an induced subgraph.

Claim 1. *If S is a forest, then $\Delta(S) \leq 2$. If S has a cycle, then each component of S has at most one cycle, which is a triangle. Moreover, if S has at least three components, then S has exactly one cycle, which is a triangle.*

Proof. If S is a forest, then, since G_3 is $K_{1,3}$-free and contains S as an induced subgraph, we have $\Delta(S) \leq 2$. If S has a cycle, then, since the only common induced cycle of G_3 and G_8 is a triangle, any induced cycle of S should be a triangle. In G_3, each pair of disjoint triangles are joined by a path of length at most two, while in G_8, the distance between the two triangles is three. Hence no component of S can contain two triangles, i.e., each component of S has at most one cycle, which is a triangle. Since G_3 contains no induced subgraph with two triangles and with at least three components, S has exactly one cycle - a triangle - when $\mathrm{nc}(S) \geq 3$. $\square$

Since G_3 is $5K_1$-free and S is an induced subgraph of G_3, S is $5K_1$-free and hence $\mathrm{nc}(S) \leq 4$. If S is connected, then S is an induced subgraph of $P_7, B_{1,4}$ or $N_{1,1,3}$ by Theorem C. Hence we assume that $2 \leq \mathrm{nc}(S) \leq 4$ and we need to consider the following three possibilities.

Subcase 1.1: $\mathrm{nc}(S) = 2$.

If S has no cycle, then $\Delta(S) \leq 2$ by Claim 1, and since all maximal induced subgraphs of G_3 with maximum degree at most two and exactly two components are $P_6 \cup K_1$ and $P_3 \cup P_4$, S is an induced subgraph of $P_6 \cup K_1$ or $P_3 \cup P_4$. If S has a cycle, then, by Claim 1, each component of S has at most one cycle - a triangle. If each component of S contains exactly one triangle, then, since the maximal common induced subgraph of G_3 and G_7 is $K_3 \cup Z_1$, S is an induced subgraph of $K_3 \cup Z_1$. Now, if one component of S contains exactly one triangle and the other component of S is a path, then, since all maximal common induced subgraphs of G_3 and G_7 are $Z_4 \cup K_1, Z_1 \cup P_4, N_{1,1,1} \cup K_2$ or $B_{1,2} \cup K_1$, S is an induced subgraph of some of them.

Observing that $P_6 \cup K_1$ is an induced subgraph of $Z_4 \cup K_1$, and that $P_3 \cup P_4$ is an induced subgraph of $Z_1 \cup P_4$, we summarize that S is an induced subgraph of $K_3 \cup Z_1, Z_4 \cup K_1, Z_1 \cup P_4, N_{1,1,1} \cup K_2$ or $B_{1,2} \cup K_1$.

Subcase 1.2: $\mathrm{nc}(S) = 3$.

If S has no cycle, then $\Delta(S) \leq 2$ by Claim 1, and since the only maximal induced subgraph of G_3 with maximum degree at most two and exactly three components is $P_4 \cup K_2 \cup K_1$, S is an induced subgraph of $P_4 \cup K_2 \cup K_1$. If S has a cycle, then by Claim 1, S has exactly one cycle - a triangle. Then, all the maximal induced subgraphs in G_3 with exactly three components containing exactly one triangle are $Z_2 \cup 2K_1, Z_1 \cup K_1 \cup K_2$ and $K_3 \cup P_4 \cup K_1$, so S is an induced subgraph of $Z_2 \cup 2K_1, Z_1 \cup K_1 \cup K_2$ or $K_3 \cup P_4 \cup K_1$.

Observing that $P_4 \cup K_2 \cup K_1$ is an induced subgraph of $K_3 \cup P_4 \cup K_1$, and that $Z_2 \cup 2K_1$ as well as $Z_1 \cup K_2 \cup K_1$ are induced subgraphs of $Z_4 \cup K_1$, we summarize that S is an induced subgraph of $K_3 \cup P_4 \cup K_1$ or $Z_4 \cup K_1$ (which is already mentioned in the previous subcase).

Subcase 1.3: $\mathrm{nc}(S) = 4$.

If S has no cycle, then $\Delta(S) \leq 2$ by Claim 1, and since the only maximal induced subgraph of G_3 with maximum degree at most two and exactly four components is $2K_2 \cup 2K_1$, S is an induced subgraph of $2K_2 \cup 2K_1$. If S has a cycle, then by Claim 1, S has exactly one cycle - a triangle. Then the maximal induced subgraph containing exactly one triangle in G_3 with exactly four components is $K_3 \cup K_2 \cup 2K_1$, so S is an induced subgraph of $K_3 \cup K_2 \cup 2K_1$. Since $2K_2 \cup 2K_1$ is an induced subgraph of $K_3 \cup K_2 \cup 2K_1$, and $K_3 \cup K_2 \cup 2K_1$ is an induced subgraph of $K_3 \cup P_4 \cup K_1$, S is an induced subgraph of $K_3 \cup P_4 \cup K_1$ (which is already mentioned in the previous subcase).

Summarizing all possibilities in Case 1, we get that S is an induced subgraph of one of the graphs in $\{K_3 \cup Z_1, Z_1 \cup P_4, Z_4 \cup K_1, N_{1,1,1} \cup K_2, B_{1,2} \cup K_1, K_3 \cup P_4 \cup K_1\}$.

<u>Case 2:</u> $R = K_{1,4}$.

Each of the graphs G_3, G_4, G_6, G_9 is $K_{1,4}$-free, hence each of them contains S as an induced subgraph. Note that G_6 is K_3-free. Since S is not an induced subgraph of P_3 or $4K_1$, considering G_4, S is an induced subgraph of some of $C_4, C_5, P_4, S_{1,1,3}, P_3 \cup 2K_1, P_3 \cup K_2, 3K_1 \cup K_2$, where $S_{1,1,3}$ denotes the graph obtained from $K_{1,3}$ by subdividing one edge twice. Since G_3 is $\{C_4, C_5, K_{1,3}\}$-free and G_9 is $P_3 \cup K_2$-free, it remains that S is an induced subgraph of $P_3 \cup 2K_1$ or $3K_1 \cup K_2$.

<u>Case 3:</u> $R = K_{1,k}$ *with* $k \geq 5$.

Each of the graphs G_3, G_4, G_5, G_6, G_9 is $K_{1,5}$-free, hence each of them contains S as an induced subgraph. Note that G_6 is K_3-free. Since S is not an induced subgraph of P_3 or $4K_1$, considering G_4, S is an induced subgraph of some of $C_4, C_5, P_4, S_{1,1,3}, P_3 \cup 2K_1, P_3 \cup K_2, 3K_1 \cup K_2$. Since G_3 is $\{C_4, C_5, K_{1,3}\}$-free, G_5 is $\{P_4, P_3 \cup K_1\}$-free and G_9 is $P_3 \cup K_2$-free, it remains that S an induced subgraph of $3K_1 \cup K_2$.

<u>Case 4:</u> $R = kK_1$ *with* $k \geq 5$.

For each $i \in \{3, 4, 5\}$, G_i is $5K_1$-free, hence each of them contains S as an induced subgraph. Therefore, S is also $5K_1$-free, implying that $\mathrm{nc}(S) \leq 4$. If $\mathrm{nc}(S) = 1$, then, since the maximal common induced subgraph of G_3, G_4 and G_5 is L_l with $l \geq 3$, S is an induced subgraph of L_l with $l \geq 3$. If $2 \leq \mathrm{nc}(S) \leq 4$, then, since the maximum induced subgraph of G_5 is $3K_1 \cup K_l$ with $l \geq 2$, S is an induced subgraph of $3K_1 \cup K_l$ with $l \geq 2$. ■

Proof of Theorem 3.

Sufficiency follows from Theorem O and Theorems 11, 12, 13 and 14. Hence it remains to show necessity.

Let R, S be a pair of graphs of order at least three other than P_3 and $3K_1$. Consider the graphs $G_1, \dots, G_9$ shown in Fig. 3 and G_{10}, G_{11}, G_{12} shown in Fig. 4. For $1 \leq i \leq 12$, G_i is non-2-factorable of arbitrarily large order and hence it contains at least one of R, S as an induced subgraph.

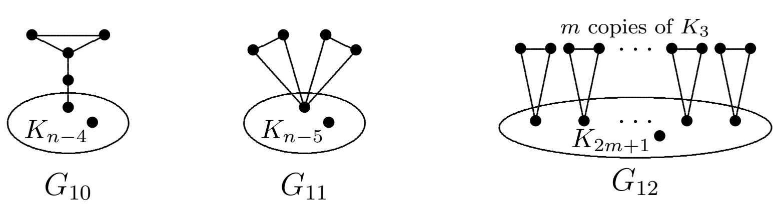

Figure 4: Connected non-2-factorable graphs with minimum degree 2 of arbitrarily large order

We now show that either R or S is edgeless or a star. Suppose, to the contrary, that neither R nor S is edgeless or a star, and recall that neither R nor S is an induced subgraph of P_3

or $4K_1$. If, say, $|V(R)| \leq 3$, then R is K_3 or $K_1 \cup K_2$, and if $|V(R)| \geq 4$, then R contains an induced $K_1 \cup K_2$ when R is disconnected or a tree, or any induced cycle in R contains an induced K_3, C_4 or a $K_1 \cup K_2$. Thus, in any case, the graph R (and symmetrically also S) contains some of $K_3, C_4, K_1 \cup K_2$ as an induced subgraph. We may assume, without loss of generality, that R is an induced subgraph of G_1. Since G_1 is $\{K_3, K_1 \cup K_2\}$-free, R contains C_4 as an induced subgraph. Since G_2 is C_4-free, G_2 contains S as an induced subgraph, and since G_2 is $\{C_4, K_1 \cup K_2\}$-free, S contains K_3 as an induced subgraph. But then G_6 is $\{K_3, C_4\}$-free, implying that G_6 is $\{R, S\}$-free and hence it has a 2-factor, a contradiction.

In the rest of the proof we assume (up to a symmetry) that R is edgeless or a star. We now consider the following four cases.

Case 1: $R = K_{1,3}$.

Since $\alpha(G_{10}) = 3$ and S is an induced subgraph of G_{11}, S is $4K_1$-free and hence $\mathrm{nc}(S) \leq 3$. If S is connected, then S is an induced subgraph of Z_2 by Theorem D. Hence we assume that $2 \leq \mathrm{nc}(S) \leq 3$.

Claim 1. *If S is a forest, then $\Delta(S) \leq 2$. If S has a cycle, then S has only one cycle, which is a triangle.*

Proof. If S is a forest, then, since G_3 is $K_{1,3}$-free and contains S as an induced subgraph, we have $\Delta(S) \leq 2$. If S has a cycle, then, since the only common induced cycle of G_8 and G_{12} is a triangle, and G_{12} does not contain two vertex disjoint cycles as an induced subgraph, S contains only one cycle - a triangle. □

Suppose first that $\mathrm{nc}(S) = 2$. If S is a forest, then $\Delta(S) \leq 2$ by Claim 1. Since the maximal induced forest in G_{10} with maximum degree at most two and exactly two components is $P_3 \cup K_2$, S is an induced subgraph of $P_3 \cup K_2$. If S has a cycle, then, by Claim 1, S has only one cycle - a triangle, and considering G_{10}, we observe that S is an induced subgraph of $Z_1 \cup K_2$.

Now suppose that $\mathrm{nc}(S) = 3$. If S is a forest, then $\Delta(S) \leq 2$ by Claim 1. Since the maximal induced forest in G_{10} with maximum degree at most two and exactly three components is $K_1 \cup 2K_2$, S is an induced subgraph of $K_1 \cup 2K_2$. If S has a cycle - a triangle, considering G_{10}, we observe that S is an induced subgraph of $K_1 \cup K_2 \cup K_3$.

Note that $K_1 \cup 2K_2$ is an induced subgraph of $K_1 \cup K_2 \cup K_3$. Summarizing all possibilities, we conclude that S is an induced subgraph of $P_3 \cup K_2, Z_1 \cup K_2$ or $K_1 \cup K_2 \cup K_3$.

Case 2: $R = K_{1,k}$ *with* $k \geq 4$.

Each of the graphs G_6, G_9, G_{10} and G_{11} is $K_{1,4}$-free, hence each of them contains S as an induced subgraph. Since any common induced subgraph of G_6 and G_{10} is a forest with maximum degree at most two, S is a forest with $\Delta(S) \leq 2$. If $\mathrm{nc}(S) = 2$, then, since the maximal common induced subgraph of G_9 and G_{11} with maximum degree at most two and exactly two components is $K_1 \cup K_2$, S is an induced subgraph of $K_1 \cup K_2$. If $\mathrm{nc}(S) = 3$, then, since the maximal common induced subgraph of G_9 and G_{11} with maximum degree at most two and exactly three components is $2K_1 \cup K_2$, S is an induced subgraph of $2K_1 \cup K_2$. Clearly, $K_1 \cup K_2$ is an induced subgraph of $2K_1 \cup K_2$, hence we conclude that S is an induced subgraph of $2K_1 \cup K_2$.

Case 3: $R = kK_1$ *with* $k = 4$.

For each $i \in \{10, 11\}$, G_i is $4K_1$-free and hence it contains S as an induced subgraph. Therefore, S is also $4K_1$-free, implying that $\mathrm{nc}(S) \leq 3$. If $\mathrm{nc}(S) = 1$, then, since the maximal common induced subgraph of G_{10} and G_{11} is L_l with $l \geq 3$, S is an induced subgraph of L_l with $l \geq 3$. If $2 \leq \mathrm{nc}(S) \leq 3$, then, since the maximal common induced subgraph of G_{10} and G_{11} is $K_1 \cup K_2 \cup K_l$ with $l \geq 2$, S is an induced subgraph of $K_1 \cup K_2 \cup K_l$ with $l \geq 2$.

Case 4: $R = kK_1$ *with* $k \geq 5$.

For each $i \in \{5, 10, 11\}$, G_i is $4K_1$-free and hence it contains S as an induced subgraph. Therefore, S is also $4K_1$-free, implying that $\mathrm{nc}(S) \leq 3$. If $\mathrm{nc}(S) = 1$, then, since the maximal common induced subgraph of G_{10} and G_{11} is L_l with $l \geq 3$, S is an induced subgraph of L_l with $l \geq 3$. If $2 \leq \mathrm{nc}(S) \leq 3$, then, since the largest common induced subgraph of G_5 and G_{11} is $2K_1 \cup K_l$ with $l \geq 3$, S is an induced subgraph of $2K_1 \cup K_l$ with $l \geq 3$. ■

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