FORBIDDEN SUBGRAPHS AND PANCYCLICITY

Ralph Faudree [1,2]
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
U. S. A

Ronald Gould [1]
Department of Mathematics
Emory University
Atlanta, GA 30322
U. S. A.

Zdeněk Ryjáček
Department of Mathematics
University of West Bohemia
30614 Pilsen
Csech Republic

Ingo Schiermeyer [3]
Lehrstuhl C für Mathematik
Technische Hochschule Aachen
D-52056 Aachen
Germany

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Abstract

We prove that every 2–connected $K_{1,3}$-free and Z_3−free graph is hamiltonian except for two graphs. Furthermore, we give a complete characterization of all 2−connected, $K_{1,3}$-free graphs, which are not pancyclic, and which are Z_3-free, B-free, W-free, or HP_7−free.

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1 INTRODUCTION

We only consider simple graphs and refer to [BM] for terminology and notation not defined here. A graph G with $n \geq 3$ vertices is *hamiltonian* if G contains a cycle of length n and *pancyclic* if G contains a cycle C_k of length k for each k with $3 \leq k \leq n$.

If C_m is a cycle with m vertices labeled $v_1, v_2, \cdots, v_m$ such that $\{v_i v_{i+1} | 1 \leq i \leq m-1\} \cup \{v_m v_1\} \subset E(G)$ and $v_j v_{j+k} \in E(G)$ for some j, k (modulo m), then the edge $v_j v_{j+k}$ is called a *k–chord* of C_m. Clearly, this $k-$chord can be used to construct a cycle of length $m - k + 1$ from the given cycle C_m. If G and G' are graphs, then we say that G is *G'-free* if G contains no induced subgraph isomorphic to G'. Specifically, we denote by C the claw $K_{1,3}$, by B the bull, by W the wounded, by D the deer, by N the net, by H the hourglass, by P_k and C_k the path and the cycle on k vertices, and by Z_k the graph obtained by identifying a vertex of K_3 with an end–vertex of P_{k+1} (see Figure 1).

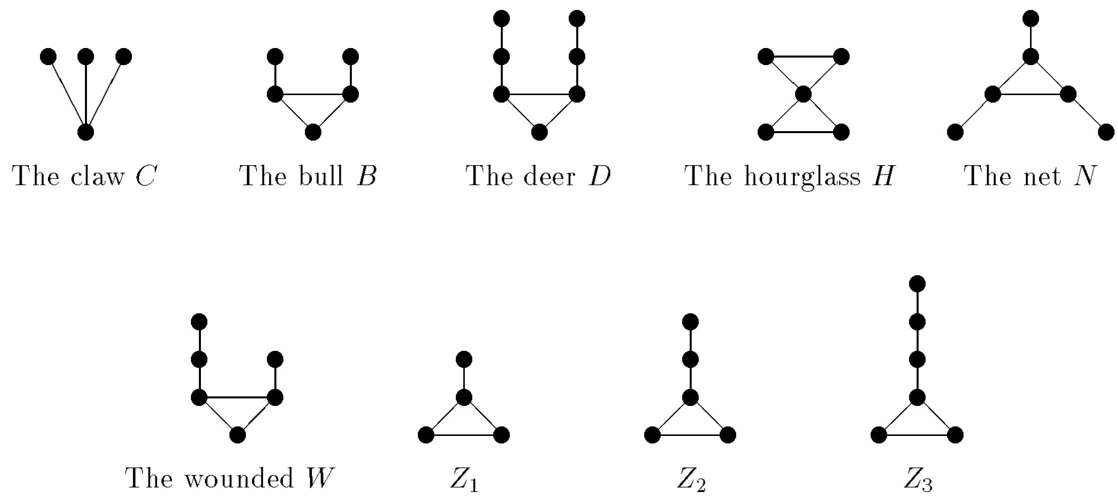


Figure 1

Bedrossian [Be] characterized all pairs of forbidden subgraphs for hamiltonian graphs.

Theorem A1 [Be]. Let R and S be connected graphs ($R, S \not\cong P_3$) and G be a 2-

connected graph. Then G is RS-free implies G is hamiltonian if, and only if, $R \cong C$ and S is one of the graphs $C_3, P_4, P_5, P_6, Z_1, Z_2, B, N$, or W.

Bedrossian [Be] also characterized all pairs of forbidden subgraphs for pancyclic graphs.

THEOREM A2 [Be]. Let R and S be connected graphs $(R, S \neq P_3)$ and let G $(G \neq C_n)$ be a 2-connected graph. Then G is $RS-$ free implies G is pancyclic if, and only if, $R \cong C$ and S is one of the graphs P_4, P_5, Z_1, or Z_2.

Bondy [Bo] proposed the following metaconjecture.

METACONJECTURE A3 [Bo]. Almost any non-trivial condition on a graph which implies that the graph is hamiltonian also implies that the graph is pancyclic. (There may be a simple family of exceptional graphs.)

Although the Metaconjecture is not true in general, it holds for a remarkably large number of sufficient conditions for hamiltonian graphs. In this paper we will examine Theorem A1 in light of the metaconjecture. The following (additional) results have already been established.

THEOREM A4 [BV]. If G is a 2-connected CDP_7-free graph, then G is hamiltonian.

THEOREM A5 [FRS]. Let G be a 2-connected, $C-$free graph. If, moreover, G is DP_7- free and $n \geq 14, G$ is P_6- free and $n \geq 10$, or G is P_5- free and $n \geq 6$, then G is pancyclic.

REMARK. There are 50 exceptional graphs, which are CDP_7- free and not pancyclic.

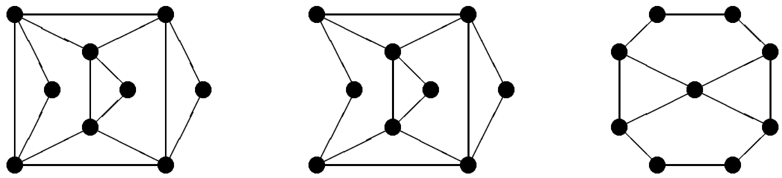
THEOREM A6 [FRS]. Let G be a 2-connected, C-free graph. If G is HP_7- free, then G is hamiltonian.

THEOREM A7 [FRS]. Let G be a 2-connected, C-free graph. If, moreover, G is HP_7- free and $n \geq 9$, then G is either pancyclic or missing only one cycle length.

2 RESULTS

In order for the reader to more easily follow the development of the results of this paper, we shall state and discuss the results in this section and hold the proofs until the next section. In [FRS] the graph H_1 (see Figure 2) shows that C and Z_3, as forbidden subgraphs, are not sufficient to guarantee even hamiltonicity. The natural question, whether there exists an infinite class of exceptional graphs or not, led to our first Theorem.

THEOREM B1. If G is a 2-connected CZ_3-free graph, then G is either hamiltonian or isomorphic to H_1 or H_2. (see Figure 2).



H_1 H_2 H_3

Figure 2

Next we will derive a full characterization of all 2-connected CZ_3- free graphs, which are not pancyclic.

For $r \geq 2$ let E_{4r} be the (unique) graph on $n = 4r$ vertices labeled $v_0, v_1, \cdots, v_{4r-1}$ with edge set $E(E_{4r}) = \{v_i v_{i+1} \mid 0 \leq i \leq 4r-1\} \cup \{v_{4j-4}v_{4j}, v_{4j-4}v_{4j+1}, v_{4j-3}v_{4j}, v_{4j-3}v_{4j+1} \mid 1 \leq j \leq r\}$ (indices modulo $4r$).

PROPOSITION B2 (Reduction Procedure RP). Let G be a 2-connected CZ_3-free graph on $n \geq k \geq 6$ vertices. If G contains a C_k with a chord, then G also contains a C_{k-1} or G contains a subgraph E_{4r} with $k = 4r$.

PROPOSITION B3 (Reduction Procedure RP). Let G be a 2-connected CZ_3-free graph on $n > k \geq 6$ vertices. If G contains a C_k, then G also contains a C_{k-1} and a C_3.

Proposition B4. Let G be a 2-connected CZ_3-free graph on $n \geq 5$ vertices. If G contains an induced C_5 and has no C_4, then G is isomorphic to $C_5, G_{6.1}, G_{7.2}$ or H_3 (see Figure 2 and Figure 3).

For $r \geq 2$ let F_{4r} be the (unique) graph on $n = 4r$ vertices labeled $v_1, v_2, \cdots, v_{2r}, u_1, \cdots, u_{2r}$ and with edge set $E(F_{4r}) = \{v_i v_j \mid 1 \leq i < j \leq 2r\} \cup \{u_{2i-1}u_{2i} \mid 1 \leq i \leq r\} \cup \{v_j u_j \mid 1 \leq j \leq 2r\}$.

Proposition B5. Let G be a graph on $n = 4r$ vertices for some $r \geq 2$. If G is $C-$free, $E_{4r} \subseteq G$ and G has no C_{n-1} then $G \cong F_{4r}$.

We are now ready to state our second Theorem.

Theorem B6. If G is a 2–connected CZ_3-free graph, then G is either pancyclic or belongs to one of the following three classes of exceptional graphs $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$, where (see Figure 2 and Figure 3)

$$\begin{aligned}
\mathcal{G}_1 &= \{C_n \mid n \geq 4\}\\
\mathcal{G}_2 &= \{F_{4r} \mid r \geq 2\}\\
\mathcal{G}_3 &= \{H_1, H_2, H_3, G_{6.1}, G_{7.2}\}.
\end{aligned}$$

Next we complete the characterization of all 2-connected CHP_7-free graphs that are not pancyclic, which was started in [FRS]. The proof of Theorem $A7$ (Theorem 15 in [FRS]) shows that every 2-connected CHP_7-free graph on $n \geq 9$ vertices contains all cycles from C_8 up to C_n or is missing only one cycle. Moreover, if G is missing a cycle C_k, then $k = 4r - 1$ for some $r \geq 2$ and $E_{4r} \subseteq G$.

Proposition B7. Let G be a 2-connected CHP_7-free graph on $n > 4r \geq 8$ vertices. If F_{4r} is an induced subgraph of G, then G has cycles C_k for $3 \leq k \leq 4r$.

Proposition B8. If G is a 2-connected CHP_7-free graph on $n \leq 12$ vertices, then G is either pancyclic or isomorphic to one of the following graphs: $C_4, C_5, C_6, G_{6.1}, \cdots, G_{12}$ (see Figure 3).

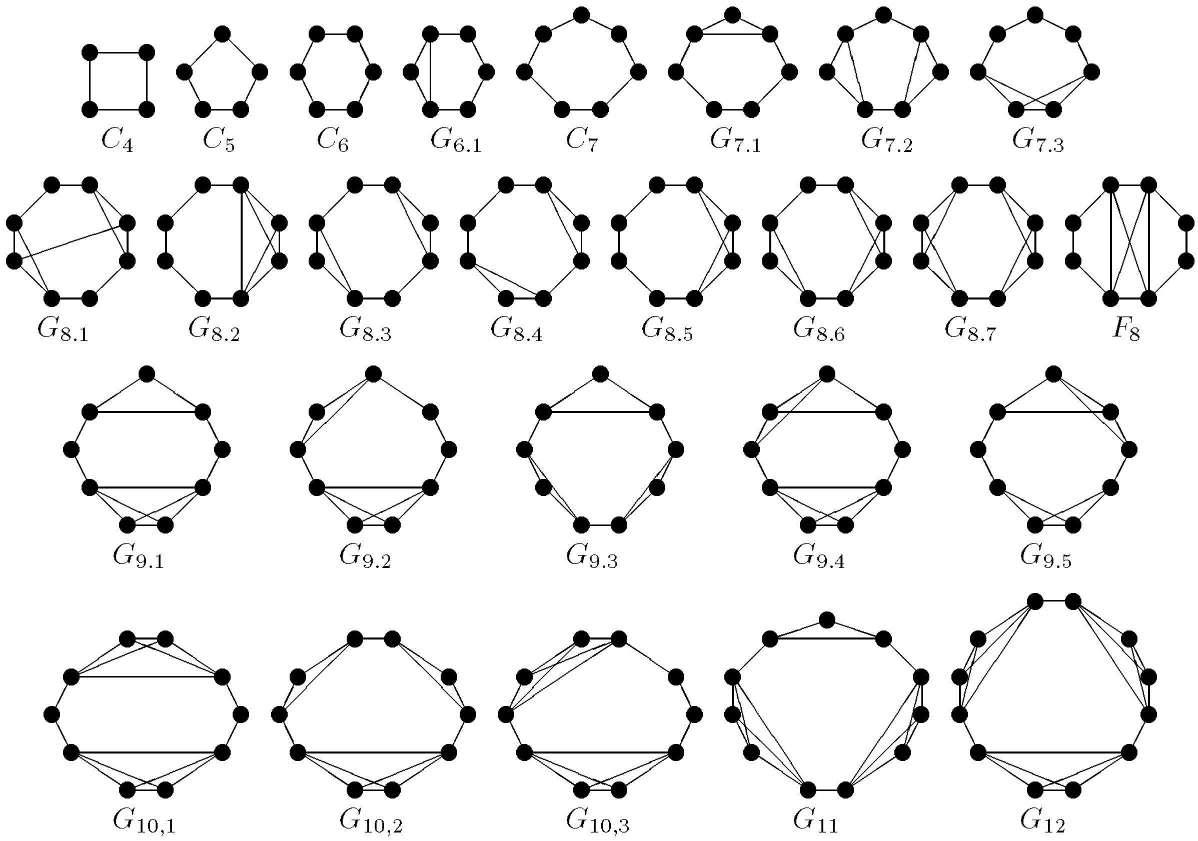


Figure 3

PROPOSITION B9. Let G be a CP_7-free graph on $n \geq 9$ vertices. If G has a hamiltonian cycle without 2-chords, then G is pancyclic.

We are now ready to present our third Theorem.

THEOREM B10. If G is a 2-connected CHP_7-free graph, then G is either pancyclic or belongs to one of the following two classes of exceptional graphs $\mathcal{G}_1 \cup \mathcal{G}_2$, where

$\mathcal{G}_1 = \{F_{4r} \mid r \geq 2\}$,

$\mathcal{G}_2 = \{C_4, C_5, C_6, G_{6.1}, \cdots, G_{12}\}$ (see Figure 3).

Next we will derive a full characterization of all 2-connected $CW-$free graphs, which are not pancyclic.

PROPOSITION B11. Let G be a $CW-$free graph on $n \geq k \geq 4$ vertices. If G contains a C_k with a chord, then G also contains a C_{k-1}.

We will now study the structure of 2-connected $CW-$free graphs which have an induced cycle C_k for some $k \geq 6$, but no cycle C_{k-1}.

Claim B12. Let G be a 2-connected $CW-$free graph on $n \geq k \geq 6$ vertices. If G has an induced cycle C of length k, then for every vertex $x \in V(G) - V(C)$ we have $N_C(x) = \{v^-, v, v^+\}$ for some vertex $v \in V(C)$.

Claim B13. If C is an induced cycle of length $k \geq 6$ in a 2-connected $CW-$free graph G, then for any two components H_1, H_2 in $G - C$ and any two vertices $x_1 \in V(H_1)$ and $x_2 \in V(H_2)$ we have $|N_C(x_1) \cap N_C(x_2)| \leq 1$.

Inspired by Claim $B12$ and Claim $B13$ we introduce the following class $\mathcal{C}_{\mathcal{C}}$ of graphs.

Let $\mathcal{C}_C$ be the class of all graphs that can be generated from all induced cycles $C_k, k \geq 4$, by replacing every vertex of C_k by a clique and joining all vertices of two cliques if and only if the corresponding vertices are adjacent in C_k. Now let G be a graph of $\mathcal{C}_C$ generated from a C_k with vertices labeled $v_1, v_2, \cdots, v_k$ and corresponding cliques K_i, $1 \leq i \leq k$. We call a subgraph $G[K_p, K_{p+1}, \cdots, K_q]$ a *saussage* if $|V(K_p)| = |V(K_q)| = 1$ and $|V(K_i)| \geq 2$ for $p+1 \leq i \leq q-1$ (indices modulo k), and G a *saussage-graph* if it has at least one saussage. Now observe that G has exactly one induced cycle of length at least 4, namely C_k, from which it has been generated. All other cycles can only occur in the saussages of G. Now for each graph $G \in \mathcal{C}_C$ let $\lambda(G)$ denote the length of the only induced cycle of length at least 4, (i.e., $\lambda(G) = k$), and let $\mu(G)$ be the maximum number of vertices among all saussages of G. Then $\lambda(G) \geq \mu(G) + 2$ if and only if G has no C_{k-1}.

Claim B14. Let G be a 2-connected $CW-$free graph on $n \geq k \geq 6$ vertices. If G has an induced C_k and no C_{k-1}, then $G \in \mathcal{C}_C$.

THEOREM B15. If G is a 2-connected $CW-$free graph, then G is either pancyclic or $G \in \mathcal{C}_C$ for some induced cycle C_k with $k \geq 4$ and $\lambda(G) \geq \mu(G) + 2$ or $G \cong G_{6.1}$ (see Figure 3).

COROLLARY B16. If G is a 2-connected CB-free graph, then G is either pancyclic or $G \in \mathcal{C}_C$ for some induced cycle C_k with $k \geq 4$ and $\lambda(G) \geq \mu(G) + 2$.

3 PROOFS

We first introduce some additonal notation which will be useful in the proofs that follow. Let C be a cycle in a graph. If an orientation of C is fixed and $u, v \in V(C)$, then by $u \overrightarrow{C} v$ we denote the consecutive vertices on C from u to v in the direction specified by the orientation of C. The same vertices, in reverse order, are given by $v \overleftarrow{C} u$. If C is a cycle of G with a fixed orientation and $u \in V(C)$, then u^+ denotes the successor of u on C and u^- its predecessor with respect to the given orientation, respectively.

Proof of Theorem B1. Suppose G satisfies the hypothesis of the theorem, but G is nonhamiltonian. Let C be a longest cycle of G with a fixed orientation. Since G is 2-connected, there exists a path of length at least 2, internally–disjoint with C, that connects two vertices of C. Let $P = v_1 u_1 u_2 \cdots u_r v_2$ be such a path of minimum length, implying that P is an induced path unless $v_1 v_2 \in E(G)$. For $i = 1, 2$, let w_i be the first vertex in $v_i^+ \overrightarrow{C} v_{3-i}$ satisfying $w_i v_i \notin E(G)$ (existing by Lemma 2 in [BV]). Since G is C−free, $v_i^- v_i^+ \in E(G)$ for $i = 1, 2$. Hence, since C is a longest cycle, $|V(v_i^+ \overrightarrow{C} v_{3-i}^-)| \geq 3$ for $i = 1, 2$.

Case 1. Suppose $\{v_1^{--} v_1, v_1 v_1^{++}, v_2^{--} v_2, v_2 v_2^{++}\} \cap E(G) \neq \emptyset$.

Without loss of generality we may assume that $v_1 v_1^{++} \in E(G)$. For $i = 1, 2$ let x_i be an arbitrary vertex in $v_i^+ \overrightarrow{C} w_i$ and u be a vertex in $V(P) - \{v_1, v_2\}$. Then $ux_1, ux_2, x_1 v_2, x_2 v_1, x_1 x_2 \notin E(G)$ (by Lemma 2 in [BV]). If $v_1 v_2 \in E(G)$, then $G[\{v_1^{++}, v_1^+, v_1, v_2, w_2^-, w_2\}]$ is an induced Z_3, a contradiction. If $v_1 v_2 \notin E(G)$, then $G[\{v_1^{++}, v_1^+, v_1, u_1, \cdots, u_r, v_2, w_2^-, w_2\}]$ is an induced Z_{r+3}, a contradiction, since $r \geq 1$ and G is Z_3−free.

Case 2. $\{v_i^{--} v_i, v_i v_i^{++}\} \notin E(G)$ for $i = 1, 2$.

If $v_1 v_2 \notin E(G)$ and $r \geq 2$, then $G[\{v_1^-, v_1, v_1^+, u_1, \cdots, u_r, v_2\}]$ is an induced Z_{r+1}, a contradiction, since G is Z_3−free. If $v_1 v_2 \in E(G)$, then $r = 1$, since otherwise $G[\{v_1^-, v_1, v_2, u_1\}]$

would be an induced claw. Hence we may assume that $r=1$. With $|V(v_i^+ \overrightarrow{C} v_{3-i}^-)| \geq 3$ for $i=1,2$ we have $n \geq 9$. If $n=9$ then $v_i^+ v_{3-i}^- \in E(G)$ for $i=1,2$, since G is Z_3-free. Now observe that G is CZ_3-free and that (as above) no other edges are possible, since C is a longest cycle. Hence, if $n=9$, then G is either hamiltonian or isomorphic to H_1 or H_2. If $n \geq 10$, then we may assume without loss of generality that $|V(v_1^+ \overrightarrow{C} v_2^-)| \geq 4$. We now consider $\{v_1^-, v_1, v_1^+, v_2^{--}, v_2^-, v_2\}$ if $v_1v_2 \in E(G)$ and $\{v_1^-, v_1, v_1^+, u_1, v_2, v_2^-, v_2^{--}\}$ if $v_1v_2 \notin E(G)$, respectively. Then $v_1^+v_2^- \in E(G)$ or $v_1^+v_2^{--} \in E(G)$, since otherwise $G[\{v_1^-, v_1, v_1^+, v_2^{--}, v_2^-, v_2\}]$ would be an induced Z_3 and $G[\{v_1^-, v_1, v_1^+, u_1, v_2, v_2^-, v_2^{--}\}]$ would be an induced Z_4, respectively. If $v_1^+v_2^- \in E(G)$, then $v_1^{++}v_2^-, v_1^+v_2^{--} \in E(G)$, since G is $C-$free and $v_1v_1^{++}, v_1v_2^-, v_1^+v_2, v_2^{--}v_2 \notin E(G)$. Now considering the claw $\{v_1, v_1^+, v_1^{++}, v_2^{--}\}$ we conclude that $v_1^{++}v_2^{--} \in E(G)$. If $v_1^+v_2^- \notin E(G)$, then $v_1^{++}v_2^-, v_1^+v_2^{--} \in E(G)$, since G is Z_3-free (symmetric argument). Again we conclude that $v_1^{++}v_2^{--} \in E(G)$. Now $v_2^{--}v_2^+ \notin E(G)$, since otherwise $v_2^{--}v_2^+ \overrightarrow{C} v_1^- v_1^+ v_1 v_2 v_2^- v_1^{++} \overrightarrow{C} v_2^{--}$ or $v_2^{--}v_2^+ \overrightarrow{C} v_1^- v_1^+ v_1 \overrightarrow{P} v_2 v_2^- v_1^{++} \overrightarrow{C} v_2^{--}$ would be a cycle longer than C. But then $G[\{v_2^{--}, v_1^{++}, v_1^+, v_1, v_2, v_2^+\}]$ is an induced Z_3 when $v_1v_2 \in E(G)$, and $G[\{v_2^{--}, v_1^{++}, v_1^+, v_1, u_1, v_2, v_2^+\}]$ is an induced Z_4 when $v_1v_2 \notin E(G)$, respectively, a contradiction. ■

For the proof of Proposition B2, the following four statements for $C-$free graphs can easily be verified and will be frequently used and just referenced by the indicated label.

(A) Let C_m be a cycle with $m \geq 2k+2 \geq 6$ vertices labeled $v_1, v_2, \cdots, v_m$ and a $k-$chord v_jv_{j+k}. If there are no $i-$chords for $2 \leq i \leq k-1$, then $v_{j-1}v_{j+k}, v_jv_{j+k+1} \in E(G)$.

(B) If, moreover, $v_{j-1}v_{j+k-1} \notin E(G)$ or $v_{j+1}v_{j+k+1} \notin E(G)$, then $v_{j-1}v_{j+k+1} \in E(G)$.

(C) Let v_jv_{j+i} be an $i-$chord with $3 \leq i \leq \frac{k}{2}$ in a cycle C_k without $2-$chords. If $v_jv_{j+i-1} \notin E(G)$, then $v_jv_{j+i+1} \in E(G)$, and likewise if $v_{j+1}v_{j+i} \notin E(G)$, then $v_{j-1}v_{j+i} \in E(G)$.

(D) Let v_jv_{j+i} be an $i-$chord in a cycle C_k. If $i \geq 2$ and $v_{j+1}v_{j+i+2} \in E(G)$ or if $i \geq 3$ and $v_{j+2}v_{j+i+1} \in E(G)$, then $v_jv_{j+i} \overleftarrow{C} v_{j+1}v_{j+i+2} \overrightarrow{C} v_j$ or $v_jv_{j+i} \overleftarrow{C} v_{j+2}v_{j+i+1} \overrightarrow{C} v_j$ is a C_{k-1}, respectively.

PROOF OF PROPOSITION B2. Let $v_1, \cdots, v_k$ be the vertices of C_k and i $(2 \le i \le \frac{k}{2})$ be the smallest integer such that G has an i-chord. Among all chords of C_k choose such a minimal i-chord $(2 \le i \le \frac{k}{2})$. Choose a labeling $v_1, v_2, \cdots, v_k$ of the vertices of C_k such that $(\{v_jv_{j+1} \mid 1 \le j \le k-1\} \cup \{v_kv_1, v_1v_{i+1}\}) \subset E(G)$. We now distinguish the following three cases.

Case 1. Suppose $i = 2$

Then $v_1v_3v_4 \cdots v_kv_1$ is a C_{k-1}.

Case 2. Suppose $i = 3$

By (A) we have $v_1v_5, v_kv_4 \in E(G)$. If $v_2v_5 \in E(G)$, then we obtain a C_{k-1} by (D). Hence we may assume that $v_2v_5 \notin E(G)$ and so $v_kv_5 \in E(G)$ by (B). If $v_4v_7 \in E(G)$, then $v_kv_5v_1v_2v_3v_4v_7\overrightarrow{C}v_k$ is a C_{k-1}. Hence we may assume that $v_4v_7 \notin E(G)$. Suppose now that $v_5v_8 \notin E(G)$. If $v_4v_8 \in E(G)$, then $v_3v_8, v_4v_9 \in E(G)$ by (A) and thus $v_3v_9 \in E(G)$ by (B), since $v_3v_7 \notin E(G)$ (or else $(v_kv_5v_4v_1v_2v_3v_7\overrightarrow{C}v_k)$). But then we obtain a C_{k-1} by $v_kv_1v_4v_5v_6v_7v_8v_3v_9\overrightarrow{C}v_k$. Hence we may assume that $v_4v_8 \notin E(G)$. Next $v_1v_6 \notin E(G)$ (or else $(v_kv_4v_3v_2v_1v_6\overrightarrow{C}v_k)$), $v_1v_7 \notin E(G)$ (or else $(v_kv_5v_4v_3v_2v_1v_7\overrightarrow{C}v_k)$), and $v_2v_7 \notin E(G)$ by (D). Now if $v_2v_8 \in E(G)$, then $v_2v_9 \in E(G)$, since G is claw–free $(\{v_2, v_7, v_8, v_9\})$, but then $v_1v_4v_5v_6v_7v_8v_2v_9\overrightarrow{C}v_1$ is a C_{k-1}. Hence we may assume that $v_2v_8 \notin E(G)$. Again the claw $\{v_1, v_2, v_4, v_8\}$ shows that $v_1v_8 \notin E(G)$. But then $G[\{v_1, v_4, v_5, v_6, v_7, v_8\}]$ is an induced Z_3, a contradiction. This shows that $v_5v_8 \in E(G)$. A repeat of these arguments (cf. also Proof of Theorem 15 in FRS]) either gives a C_{k-1} or $k = 4r$ for some $r \ge 2$. In the latter case, $v_{4i-4}v_{4i}, v_{4i-4}v_{4i+1}, v_{4i-3}v_{4i+1} \in E(G)$ for each edge $v_{4i-3}v_{4i}, 1 \le i \le r$ (indices modulo $4r$) and the C_{4r} has no other 3-chords, 4-chords, 5-chords or 6-chords.

Case 3. Suppose $i \ge 4$

We proceed as in Case 2 and obtain that $v_1v_{i+2}, v_kv_{i+1}, v_kv_{i+2} \in E(G)$. Since i is minimal, $G[\{v_{i+2}, v_k, v_1, v_2, v_3, v_4\}]$ is an induced Z_3, a contradiction. ∎

Proof of Proposition B3. Let $v_1, \cdots, v_k$ be the vertices of C_k labeled such that $(\{v_jv_{j+1} \mid 1 \leq j \leq k-1\} \cup \{v_kv_1\}) \subset E(G)$. Since $k < n$, there is a vertex $u \in V(G) - V(C_k)$ such that $N(u) \cap V(C_k) \neq \emptyset$. If C_k has a 2–chord, then we obtain a C_{k-1} and a C_3. Hence we may assume that C_k has no 2–chords. Now, if $v_i \in N(u)$ for some $v_i \in V(C_k)$, then $\{v_{i-1}, v_{i+1}\} \cap N(u) \neq \emptyset$, since G is C-free and $v_{i-1}v_{i+1}$ would be a 2-chord. Hence G has a C_3. Next, if $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\} \subseteq N(u)$ for some $v_i \in V(C_k)$, then $v_iuv_{i+3}\overrightarrow{C}_kv_i$ is a C_{k-1}. Hence, $N(u) \cap V(C_k)$ consists of pairwise disjoint pairs $\{v_j, v_{j+1}\}$ and triples $\{v_j, v_{j+1}, v_{j+2}\}$ of consecutive vertices. We distinguish these two cases.

<u>Case 1</u>. Suppose $v_4, v_5 \in N(u)$, $v_3, v_6 \notin N(u)$

If $v_j, v_{j+3} \in N(u)$ or $v_j, v_{j-3} \in N(u)$, then we (easily) obtain a C_{k-1}. Hence, $v_{j-3}, v_{j+3} \notin N(u)$ for each $v_j \in N(u)$. Thus, $v_1, v_2, v_7, v_8 \notin N(u)$. Since C_k has no 2–chords, we have $v_1v_3, v_2v_4, v_3v_5, v_4v_6, v_5v_7, v_6v_8 \notin E(G)$. If $v_1v_4 \in E(G)$, then $v_1v_4uv_5\overrightarrow{C}v_1$ is a C_{k-1}. Hence we may assume that $v_1v_4, v_5v_8 \notin E(G)$. If both $v_2v_5, v_4v_7 \in E(G)$, then $v_2v_5uv_4v_7\overrightarrow{C}v_{k-1}$ is a C_{k-1}. Hence we may assume without loss of generality that $v_4v_7 \notin E(G)$. If $v_2v_5 \in E(G)$, then $v_3v_8 \notin E(G)$, since otherwise, $v_2v_5uv_4v_3v_8\overrightarrow{C}v_2$ is a C_{k-1}. Now, if $v_4v_8 \in E(G)$, then $G[\{v_3, v_4, u, v_8\}]$ is an induced claw, a contradiction. Hence $v_4v_8 \notin E(G)$, but then $G[\{v_4, u, v_5, v_6, v_7, v_8\}]$ is an induced Z_3, a contradiction. This shows that both $v_2v_5, v_4v_7 \notin E(G)$. Hence, $v_1v_5, v_4v_8 \in E(G)$, since G is Z_3–free. Considering the claws $\{v_1, v_5, u, v_6\}$ and $\{v_3, v_4, u, v_8\}$ we conclude that $v_1v_6, v_3v_8 \in E(G)$, since G is C–free. But then $v_1v_6v_5uv_4v_3v_8\overrightarrow{C}v_1$ is a C_{k-1}.

<u>Case 2</u>. Suppose $v_2, v_3, v_4 \in N(u)$, $v_1, v_5 \notin N(u)$.

Since C_k has no 2–chords, we have $v_2v_4, v_3v_5, v_4v_6, v_5v_7 \notin E(G)$. If $v_3v_6 \in E(G)$ or $v_4v_7 \in E(G)$ or $v_3v_7 \in E(G)$, then $v_2uv_3v_6\overrightarrow{C}v_2$ or $v_3uv_4v_7\overrightarrow{C}v_3$ or $v_2uv_4v_3v_7\overrightarrow{C}v_2$ is a C_{k-1}, respectively. Hence we may assume that $v_3v_6, v_3v_7, v_4v_7 \notin E(G)$. As in Case 1 we have $uv_6, uv_7 \notin E(G)$, since $uv_3, uv_4 \in E(G)$. But then $G[\{v_3, u, v_4, v_5, v_6, v_7\}]$ is an induced Z_3, a contradiction. ∎

Proof of Proposition B4. Let $v_1, v_2, \cdots, v_5$ be the vertices of the induced C_5 such that $\{v_jv_{j+1}|1 \le j \le 4\} \cup \{v_5v_1\} \subseteq E(G)$. If $n = 5$, then $G \cong C_5$. If $n > 5$, then let $H = G - V(C_5)$. Since G is 2–connected, there is a vertex $x \in V(H)$ such that $N_{C_5}(x) \neq \emptyset$. As in the proof of Proposition B3, we conclude that x has either two or three consecutive neighbors on the C_5. Since G has no C_4 we conclude that for every $x \in N_H(V(C_5))$ we have $|N(x) \cap V(C_5)| = 2$ and $N_{C_5}(x) = \{v_i, v_{i+1}\}$ for some i. By the 2–connectedness of G we conclude that each component of H either is an isolated vertex or has at least two vertices each of them having two neighbors on C_5. Thus for $n = 6$ we obtain the unique exceptional graph $G_{6.1}$ (see Figure 3). Since G has no C_4, for $n \ge 7$ there is no pair of vertices $x, y \in V(H)$ such that $N_{C_5}(x) = N_{C_5}(y) = \{v_i, v_{i+1}\}$ for some i (which gives $(xv_iyv_{i+1}x)$). Without loss of generality we may assume $N_{C_5}(x) = \{v_1, v_2\}$ for some $x \in V(H)$. Suppose $N_{C_5}(y) = \{v_3, v_4\}$ for some $y \in V(H) - \{x\}$. Since G has no C_4, we have $xy \notin E(G)$ (or else (xv_2v_3yx)). But then $G[\{x, v_2, v_1, v_5, v_4, y\}]$ is an induced Z_3, a contradiction. Hence we may assume that $N_H(v_4) = \emptyset$ and thus $|N_H(C_5)| = 2$ (symmetric argument). Without loss of generality we may assume that $N_{C_5}(y) = \{v_5, v_1\}$ for some $y \in V(H) - \{x\}$. Since G has no C_4, we have $xy \notin E(G)$ (or else (yv_1v_2xy)). Thus for $n = 7$ we obtain the unique exceptional graph $G_{7,2}$ (see Figure 2). For $n \ge 8$ we conclude that H consists of one component, since $|N_H(C_5)| = 2$. Let $xw_1w_2 \cdots w_ry$ be a shortest path connecting x and y in H. Then $r \ge 2$, since G has no C_4 (xv_1yw_1x). If $r \ge 3$, then $G[\{v_1, v_2, x, w_1, w_2, w_3\}]$ is an induced Z_3, a contradiction. Hence we have $r = 2$. This gives the unique exceptional graph H_3 (see Figure 2). ∎

Proof of Proposition B5. Let the vertices of G be labeled $v_1, v_2, \cdots, v_n$ such that $\{v_jv_{j+1}|1 \le j \le n-1\} \cup \{v_nv_1\} \subset E(G)$, $\{v_1v_4, v_5v_8, \cdots, v_{4r-3}u_{4r}\} \subset E(G)$ and $\{v_{4i-3}v_{4i+1}, v_{4i-4}v_{4i}, v_{4i-4}v_{4i+1}\} \subset E(G)$ for each edge $v_{4i-3}v_{4i}$ (indices modulo n). If $r = 2$ then $G \cong F_8(= E_8)$, since any additional edge gives a C_7. For $r > 2$ we perform an induction on k for $1 \le k \le \lfloor \frac{r}{2} \rfloor$. For each k and all possible i-chords with $4k - 1 \le i \le 4k + 2$ (and $3 \le i \le 2r$) we shall show:

$$\begin{aligned}
i = 4k-1: &\quad v_1v_{4k} \in E(G);\quad v_2v_{4k+1}, v_3v_{4k+2}, v_nv_{4k-1} \notin E(G)\\
i = 4k: &\quad v_nv_{4k}, v_1v_{4k+1} \in E(G);\quad v_2v_{4k+2}, v_3v_{4k+3} \notin E(G)\\
i = 4k+1: &\quad v_nv_{4k+1} \in E(G);\quad v_1v_{4k+2}, v_2v_{4k+3}, v_3v_{4k+4} \notin E(G)\\
i = 4k+2: &\quad v_nv_{4k+2}, v_1v_{4k+3}, v_2v_{4k+4}, v_3v_{4k+5} \notin E(G).
\end{aligned}$$

By the cyclic structure of G these properties then remain valid for all induced subgraphs $G[\{v_{4j}, v_{4j+1}, \cdots, v_{4j+4k+5}\}]$. We first show the induction step "$k \longrightarrow k+1$". To show that $v_1v_{4k}, v_nv_{4k}, v_1v_{4k+1}, v_nv_{4k+1} \in E(G)$, we consider suitable claws and make use of the claw–freeness of G:

$$\begin{aligned}
\{v_1, v_{4k-1}, v_{4k}, v_{4k+4}\}: \quad & v_1v_{4k}, v_{4k-1}v_{4k}, v_{4k}v_{4k+4} \in E(G), \\
& v_1v_{4k-1}, v_{4k-1}v_{4k+4} \notin E(G); \\
& \text{hence} \quad v_1v_{4k+4} \in E(G). \\
\{v_1, v_{4k-1}, v_{4k}, v_{4k+5}\}: \quad & v_1v_{4k}, v_{4k-1}v_{4k}, v_{4k}v_{4k+5} \in E(G), \\
& v_1v_{4k-1}, v_{4k-1}v_{4k+5} \notin E(G); \\
& \text{hence} \quad v_1v_{4k+5} \in E(G).
\end{aligned}$$

$$\begin{aligned}
\{v_n, v_{4k-1}v_{4k}, v_{4k+5}\}: \quad & v_nv_{4k}, v_{4k-1}v_{4k}, v_{4k}v_{4k+5} \in E(G), \\
& v_nv_{4k-1}, v_{4k-1}v_{4k+5} \notin E(G); \\
& \text{hence} \quad v_nv_{4k+5} \in E(G).
\end{aligned}$$

$$\begin{aligned}
\{v_n, v_{4k-1}, v_{4k}, v_{4k+4}\}: \quad & v_nv_{4k}, v_{4k-1}v_{4k}, v_{4k}v_{4k+4} \in E(G), \\
& v_nv_{4k-1}, v_{4k-1}v_{4k+4} \notin E(G); \\
& \text{hence} \quad v_nv_{4k+4} \in E(G).
\end{aligned}$$

$$\begin{array}{lll}
v_2v_{4k+5} \notin E(G): & \text{else} & v_nv_{4k+4}\overleftarrow{C}v_2v_{4k+5}\overrightarrow{C}v_n \quad \text{is a} \quad C_{n-1} \\
v_nv_{4k+3} \notin E(G): & \text{else} & v_nv_{4k+3}\overleftarrow{C}v_1v_{4k+5}\overrightarrow{C}v_n \quad \text{is a} \quad C_{n-1} \\
v_3v_{4k+6} \notin E(G): & \text{else} & v_nv_{4k}\overrightarrow{C}v_{4k+4}v_1v_2v_3v_{4k+6}\overrightarrow{C}v_n \quad \text{is a} \quad C_{n-1} \\
v_2v_{4k+6} \notin E(G): & \text{else} & v_nv_4\overrightarrow{C}v_{4k+5}v_1v_2v_{4k+6}\overrightarrow{C}v_n \quad \text{is a} \quad C_{n-1} \\
v_3v_{4k+7} \notin E(G): & \text{else} & v_nv_4\overrightarrow{C}v_{4k+5}v_1v_2v_3v_{4k+7}\overrightarrow{C}v_n \quad \text{is a} \quad C_{n-1} \\
v_1v_{4k+6} \notin E(G): & \text{else} & v_nv_{4k+4}\overleftarrow{C}v_1v_{4k+6}\overrightarrow{C}v_n \quad \text{is a} \quad C_{n-1} \\
v_2v_{4k+7} \notin E(G): & \text{else} & v_1v_{4k+4}\overleftarrow{C}v_2v_{4k+7}\overleftarrow{C}v_{4k+5}v_{4k+9}\overrightarrow{C}v_1 \quad \text{is a} \quad C_{n-1} \\
v_3v_{4k+8} \notin E(G): & \text{else} & v_3v_{4k+8}\overleftarrow{C}v_5v_{4k+9}\overrightarrow{C}v_3 \quad \text{is a} \quad C_{n_1} \\
v_nv_{4k+6} \notin E(G): & \text{else} & v_nv_{4k+6}v_{4k+5}v_1\overrightarrow{C}v_{4k+4}v_{4k+8}\overrightarrow{C}v_n \quad \text{is a} \quad C_{n-1} \\
v_1v_{4k+7} \notin E(G): & \text{else} & v_nv_{4k+4}\overleftarrow{C}v_1v_{4k+7}\overleftarrow{C}v_{4k+5}v_{4k+9}\overrightarrow{C}v_n \quad \text{is a} \quad C_{n-1} \\
v_2v_{4k+8} \notin E(G): & \text{else} & v_2v_{4k+8}\overleftarrow{C}v_4v_{4k+9}\overrightarrow{C}v_2 \quad \text{is a} \quad C_{n-1} \\
v_3v_{4k+9} \notin E(G): & \text{else} & v_1v_{4k+8}\overleftarrow{C}v_3v_{4k+9}\overrightarrow{C}v_1 \quad \text{is a} \quad C_{n-1}.
\end{array}$$

Next we show the induction beginning with "$k = 1$". By the hypothesis we know that $v_1v_4, v_nv_4, v_1v_5, v_nv_5 \in E(G)$. For $k = 1$ the (12) constructions above (of a C_{n-1}) remain valid. Thus, $v_2v_5, v_nv_3, \cdots, v_3v_9 \notin E(G)$. ∎

Proof of Theorem B6. If G is nonhamiltonian, then G is isomorphic to either H_1 or H_2 by Theorem B1. Hence we may assume that G is hamiltonian. If G has a C_n without chords, then $G \cong C_n$. Hence we may assume that C_n has a chord. If G has no C_{n-1}, then by Proposition (B2) we have $E_{4r} \subseteq G$ with $n = 4r$, and thus by Proposition (B5) we conclude that $G \cong F_{4r}$. Hence we may assume that G has a C_{n-1}. Then by Proposition (B3) G has cycles C_k for $k = 3$ and $5 \leq k \leq n$. If G has no C_4 then G is isomorphic to $G_{6.1}$ or $G_{7.2}$ or H_3 by Proposition B4 and pancyclic otherwise. ∎

Proof of Proposition B7. Let $v_1, v_2, \cdots, v_{2r}, u_1, u_2, \cdots, u_{2r}$ be the vertices of F_{4r} such that $d_{F_{4r}}(u_i) = 2$, with $u_iv_i \in E(G)$ for $1 \leq i \leq 2r$ and $u_{2i-1}u_{2i} \in E(G)$ for $1 \leq i \leq r$. We know that F_{4r} is only missing a C_{4r-1}. Suppose there is a vertex $w \in V(G - F_{4r})$ such that $wu_i \in E(G)$ for some i with $1 \leq i \leq 2r$. We may assume that $wu_1 \in E(G)$. Since $v_1u_2 \notin E(G)$ and G is claw-free, we have $v_1w \in E(G)$ or $wu_2 \in E(G)$. Then $v_1wu_1u_2v_2v_3u_3u_4v_4 \cdots u_{2r-2}v_{2r-2}v_{2r-1}v_{2r}v_1$ or $v_1u_1wu_2v_2v_3u_3u_4v_4 \cdots u_{2r-2}v_{2r-2}v_{2r-1}v_{2r}v_1$ is a C_{4r-1}. Hence we may assume that $d_G(u_i) = 2$ for $1 \leq i \leq 2r$. Thus, there is a vertex $w \in V(G - F_{4r})$ such that $wv_i \in E(G)$ for some i with $1 \leq i \leq 2r$. We may assume that $wv_i \in E(G)$. Since G is claw-free and $wu_1, u_1u_2 \notin E(G)$, we have $wv_2 \in E(G)$. But then, $v_1wv_2u_3u_4v_4 \cdots u_{2r}v_{2r}v_1$ is the desired C_{4r-1}. ∎

Proof of Proposition B8. At first we generate all CHP_7-free graphs on $n \leq 8$ vertices, which are not pancyclic. For $4 \leq n \leq 7$ it can be easily verified that all exceptional graphs are given by the graphs in Figure 3. Next suppose there is a CHP_7-free graph on $n + 1$ vertices, $n \geq 8$, which is not pancyclic. Then by Proposition B9, it has a 2-chord. Using this 2-chord in the reduction procedure, we also obtain an exceptional graph on n vertices. Vice versa, the set of all exceptional graphs on $n+1$ vertices can be generated from the set of all exceptional graphs on n vertices as follows. Let G be a counterexample on n vertices $v_1, v_2, \cdots, v_n$ such that $(\{v_iv_{i+1} | 1 \leq i \leq n-1\} \cup \{v_nv_1\}) \subset E(G)$. We then successively replace each edge v_iv_{i+1}

of this C_n by a triangle with edges $v_iv_{i+1}, v_iv_{n+1}, v_{i+1}v_{n+1}$ if $1 \leq i \leq n-1$ and a triangle with edges $v_1v_n, v_1v_{n+1}, v_nv_{n+1}$, otherwise. Each new graph has to be checked as to whether it is CHP_7-free and not pancyclic, and whether additional edges adjacent to v_{n+1} are possible.

We now consider six cases.

<u>Case 1</u>. Suppose $n = 8$.

By the hypothesis of the proposition, the cycle C_8 contains a chord. Since G is claw-free, it contains a 2-chord or a 3-chord. Among all chords of C_8 choose an i-chord ($2 \leq i \leq 3$) such that i is minimal. Choose a labeling $v_1, v_2, \cdots, v_8$ of the vertices of C_8 such that $(\{v_jv_{j+1} \mid 1 \leq j \leq 7\} \cup \{v_8v_1, v_1v_{i+1}\}) \subset E(G)$.

<u>Case 1.1</u>. Suppose $i = 2$

Then G contains C_3, C_7 and C_8. If there is a 3-chord and a 4-chord then G is pancyclic, since a 4-chord gives a C_5 and a 3-chord gives C_4 and C_6. If there are only 4-chords and there is a pair of 2-chords and a 4-chord that are crossing, then since G is claw-free and has no 3-chord, G has a C_3, C_5, C_6, C_7 and C_8. If there is also a pair of a 2-chord and a 4-chord that are not crossing, then G is pancyclic. Otherwise we obtain the only exceptional graph $G_{8.1}$ having only 2-chords and 4-chords. If there are only 3-chords then G has C_3, C_4, C_6, C_7 and C_8. Now each pair of a 2-chord and a 3-chord, whether they are crossing or not, leads to a C_5 and thus G is pancyclic, or we obtain the exceptional graph $G_{8.2}$.

Hence we may assume that G has only 2-chords. Suppose first that there are no crossing 2-chords. Since G is P_7-free, there are at least two vertex disjoint 2-chords. Since G is H-free, any pair of 2-chords is vertex disjoint. Thus the only exceptional graphs with two 2-chords are given by $G_{8.3}$ and $G_{8.4}$.

Next suppose there are crossing 2-chords. If, for example $v_1v_3, v_2v_4, v_3v_5 \in E(G)$, then G is pancyclic. Hence we may assume that among every five successive vertices of C_8 there occur at most two 2-chords. We may assume that $v_2v_4 \in E(G)$. Hence $v_3v_5, v_8v_2 \notin E(G)$, since G is H-free. Thus we obtain the exceptional graphs $G_{8.5}, G_{8.6}$ and $G_{8.7}$.

Case 1.2. Suppose $i = 3$

The only exceptional graph in this case is $F_8 \in \mathcal{G}_1$.

For the sake of brevity, in the following four cases we list those exceptional graphs on n vertices, that have been generated from a specific exceptional graph on $n-1$ vertices.

Case 2. Suppose $n = 9$

$G_{8.1}$:

$G_{8.2}$: $G_{9.1}, G_{9.2}, G_{9.4}$

$G_{8.3}$: $G_{9.1}$

$G_{8.4}$: $G_{9.2}, G_{9.3}$

$G_{8.5}$: $G_{9.5}$

$G_{8.6}$: $G_{9.4}$

$G_{8.7}$:

$G_{8.8}$:

Case 3. Suppose $n = 10$

$G_{9.1}$: $G_{10.1}$

$G_{9.2}$: $G_{10.2}$, $G_{10.3}$

$G_{9.3}$: $G_{10.2}$

$G_{9.4}$:

$G_{9.5}$:

Case 4. Suppose $n = 11$

$G_{10.1}$:

$G_{10.2}$: G_{11}

$G_{10.3}$: G_{11}

Case 5. Suppose $n = 12$

G_{11}: G_{12}

Case 6. Suppose $n = 13$

The graph G_{12} is only missing a C_5. Replacing an edge by a triangle, we either obtain a C_5 and thus a pancyclic graph, or a graph that is not H-free. In the latter case, every additional (possible) edge gives a C_5. ■

Proof of Propositon B9. In ([Frs] Proposition 4) this was proved for the class of CDP_7-free graphs. However, the D-freeness is not needed there, hence the conclusion even holds in the class of CP_7 free graphs. ■

Proof of Theorem B10. Let G be a 2-connected CHP_7-free graph on $n \geq 3$ vertices. By Theorem A6 we know that G is hamiltonian. If $n \leq 8$ then G is either pancyclic or isomorphic to $C_4, C_5, C_6, G_{6.1}, \cdots, G_{8.7}, F_8$ by Proposition B8. If $n \geq 9$, then G contains all cycles from C_8 up to C_n or is missing only one cycle C_{4r-1} for some $r \geq 2$ and $F_{4r} \subseteq G$ as mentioned earlier. In the latter case, F_{4r} is an induced subgraph of G by Proposition B5. If $n > 4r$ then G has all cycles C_k for $3 \leq k \leq 4r$ by Proposition B7 and hence is pancyclic. Otherwise, $G \cong F_{4r}$ and hence G is not pancyclic. Hence we many assume that G contains all cycles from C_8 up to C_n. If, moreover, G has a cycle C_k for some $k \geq 9$ without 2-chords, then G is pancyclic by Proposition B9.

Hence, any exceptional graph must have $n = 8$ vertices or must have a cycle C_k with a 2-chord for some $k \geq 9$. All these exceptional graphs are given by Propostion B8. Furthermore, the proof of Proposition B8 shows that there are no exceptional graphs on $n \geq 13$ vertices. This completes the proof. ∎

Proof of Proposition B11. Let $v_1, \cdots v_k$ be the vertices of C_k. Let i be the smallest integer such that G has an i-chord. Since G is C- free we have $2 \leq i \leq \frac{k-1}{2}$. Among all chords of C_k choose a minimal i-chord $(2 \leq i \leq \frac{k-1}{2})$. Choose a labeling $v_1, v_2, \cdots, v_k$ of the vertices of C_k such that $(\{v_j v_{j+1} | 1 \leq j \leq k-1\} \cup \{v_k v_1, v_1 v_{i+1}\}) \subset E(G)$.

We now distinguish the following two cases.

Case 1. Suppose $i = 2$

Then $v_1 v_3 v_4 \cdots v_k v_1$ is a C_{k-1}.

Case 2. Suppose $i \geq 3$

For $i \geq 3$ we have $k \geq 7$. For $k = 7$ we conclude (successively) that all 3-chords are present, since G is C- free. Then $v_1 v_4 v_5 v_2 v_6 v_7 v_1$ is a C_6. For $k \geq 8$ we will show that G either has a C_{k-1} or that $G[\{v_{k-2}, v_{k-1}, v_k, v_1, v_2, v_{i+1}\}]$ is an induced W. Since G is a C-free and i is minimal we have $v_1 v_{i+2}, v_k v_{i+1} \in E(G)$. If $v_2 v_{i+2} \in E(G)$, then $v_k v_{i+1} \overleftarrow{C}\ v_2 v_{i+2} \overrightarrow{C}\ v_k$ is a C_{k-1}. Otherwise, $v_k v_{i+2} \in E(G)$, or $G[\{v_k, v_1, v_2, v_{i+2}\}]$ would be an induced claw. Since $i \geq 3$ and i is minimal we have $v_{k-2} v_k, v_{k-1} v_1, v_k v_2, v_2 v_{i+1} \notin E(G)$. If $v_{k-2} v_1 \in E(G)$ or $v_{k-1} v_2 \in E(G)$ or $v_{k-2} v_2 \in E(G)$, then $v_{k-2} v_1 \overrightarrow{C}\ v_{i+1} v_k v_{i+2} \overrightarrow{C}\ v_{k-2}$ or $v_{k-1} v_2 \overrightarrow{C}\ v_{i+1} v_1 v_{i+2} \overrightarrow{C}\ v_{k-1}$ or $v_{k-2} v_2 \overrightarrow{C}\ v_{i+1} v_1 v_k v_{i+2} \overrightarrow{C}\ v_{k-2}$ is a C_{k-1}. If $v_{k-1} v_{i+1} \in E(G)$

or $v_{k-2}v_{i+1} \in E(G)$, then $v_{k-1}v_{i+1}\overleftarrow{C}v_1v_{i+2}\overrightarrow{C}v_{k-1}$ or $v_{k-2}v_{i+1}\overleftarrow{C}v_kv_{i+2}\overrightarrow{C}v_{k-2}$ is a C_{k-1}. Otherwise, $G[\{v_{k-2}, v_{k-1}, v_k, v_1, v_2, v_{i+1}\}]$ is an induced W, a contradiction. ∎

Proof of Claim B12. For $k = n$ the assertion holds. Hence we may assume that $k < n$. Since G is 2-connected, there are two vertices $v \in V(C)$ and $x \in V(G) \backslash V(C)$ such that $vx \in E(G)$. Since G is $\mathcal{C}$-free and C_k is an induced cycle, we have $\{v^-, v^+\} \cap N_C(x) \neq \emptyset$. Suppose first, that $v^-, v \in N(x)$ and $v^{--}, v^+ \notin N(x)$. Then $v^{---}, v^{++} \notin N(x)$, since G has no C_{k-1}. But then $G[\{v^{--}, v^-, v, v^+, v^{++}, x\}]$ is an induced W, a contradiction. Hence we may assume that $v^-, v, v^+ \in N(x)$. Again, since G has no C_{k-1}, we conclude that $v^{--}, v^{++} \notin E(G)$. Now, if there is a vertex $w \in V(C) \cap N(x)$ such that $w \notin \{v^-, v, v^+\}$, then $G[\{v^-, v^+, w, x\}]$ is an induced claw, a contradiction. Next suppose there is a vertex $y \in V(G) \setminus V(C)$ such that $N_C(y) = \emptyset$. We may assume that there is a path yxw such that $w \in V(C)$ and $x \notin V(C)$. But then $G[\{y, x, v^-, v^+\}]$ is an induced claw, since C_k has no chords, a contradiciton. ∎

Proof of Claim B13. Suppose there are two components H_1, H_2 in $G - C$ and two vertices $x_1 \in V(H_1)$, $x_2 \in V(H_2)$ such that $|N_C(x_1) \cap N_C(x_2)| \geq 2$. By Claim B12 we then distinguish two cases.

Case 1. Suppose $N_C(x_1) = N_C(x_2) = \{w^-, w, w^+\}$ for a vertex $w \in V(C)$.

But then $G[\{x_1, x_2, w^+, w^{++}\}]$ is an induced claw, a contradiction.

Case 2. Suppose $N_C(x_1) = \{w^-, w, w^+\}$ and $N_C(x_2) = \{w, w^+, w^{++}\}$ for a vertex $w \in V(C)$.

But then $G[\{w^-, x_1, w^+, x_2, w^{++}, w^{+++}\}]$ is an induced W, a contradiction. ∎

Proof of Claim B14. We perform an induction on $p = |V(G) \backslash V(C)|$.

1. Induction beginning with $p = 0$.

Then $k = n$ and thus $G \cong C_n$. Hence $G \in \mathcal{C}_C$.

2. Induction step $p-1 \to p$.

Suppose that Claim B14 holds for all graphs with $|V(G)\backslash V(C)| \le p-1$ and let G be a graph with $|V(G)\backslash V(C)| = p$. Choose a vertex $x \in (V(G))\backslash V(C))$ and put $G' = G - x$. Then, the following properties hold:

1. G' is CW-free, since 'CW-freeness' is a hereditary property.
2. G' is 2-connected due to Claim B12.
3. C is an induced C_k in G'.
4. There is no cycle in G' of length $|V(C)|-1$.

Thus, by the induction hypothesis, $G' \in \mathcal{C}_C$. Let the vertices of C be labeled $y_1, y_2, \cdots, y_k$ and let K_i be the clique with $y_i \in V(K_i)$, $(1 \le i \le k)$, corresponding to the structure of the class $\mathcal{C}_C$. By Claim B12, x has exactly three neighbors on C, say y_{i-1}, y_i, y_{i+1}.

(i) If there is a vertex $z_i \in V(K_1)$ such that $xz_i \notin E(G)$, then $G[\{y_{i-1}x, z_i, y_{i-2}\}]$ is an induced claw, a contradiction.

(ii) If there is a vertex $z_{i+1} \in V(K_{i+1})$ such that $xz_{i+1} \notin E(G)$, then $G[\{x, y_i, y_{i-1}, y_{i-2}, z_{i+1}, y_{i+2}\}]$ is an induced W, a contradiction.

(iii) Symmetric to (ii) we have $xz_{i-1} \in E(G)$ for all vertices $z_{i-1} \in V(K_{i-1})$.

Thus $G \in \mathcal{C}_C$. ∎

Proof of Theorem B15. Let G be a 2-connected CW-free graph. By Theorem A1 we know that G is hamiltonian. By Proposition B11 we conclude that G is either pancyclic or has an induced cycle C_k for some $k \ge 4$. If $k \ge 6$ and G has no C_{k-1} then $G \in \mathcal{C}_C$ by Proposition B14 and necessarily $\lambda(G) \ge \mu(G) + 2$. Hence we may assume that $4 \le k \le 5$. If $k = n$, implying $G \cong C_n$, then $G \in \mathcal{C}_C$. Hence we may further assume that $k < n$. Since G is C-free there is no pair of vertices $v \in V(C), x \in V(G) - V(C)$ such that $xv \in E(G)$ and $xv^-, xv^+ \notin E(G)$. If $\{v^-, v, v^+\} \subset N_C(x)$ for a vertex $x \in V(G) - V(C)$, then G has a C_3 and a C_4 and thus is pancyclic. Hence, if G is not pancyclic, then $k = 5$ and for each vertex $x \in V(G) - V(C)$ we have $N_C(x) = \emptyset$ or $|N_C(x)| = 2$. In the latter case, $N_C(x) = \{v, v^+\}$ for a vertex $v \in V(C)$. Thus for $n = 6$ we obtain the exceptional graph $G_{6.1}$. Now for $n \geq 7$, suppose first that there are two vertices $x, y \in V(G) - V(C)$ such that $N_C(x) = \{v^-, v\}, N_C(y) = \{w^-, w\}$ for two vertices $v, w \in V(C)$. Since G is missing only a C_4, we cannot have $v = w$. If $w^- = v$ or $w^- = v^+$ then $xy \notin E(G)$, since otherwise xyw^-v^-x or xyw^-vx gives a C_4. But then $G[\{v^{---}, v^{--}, v^-, x, v, y\}]$ or $G[\{v^{--}, v^-, x, v, w^-, y\}]$ is an induced W, a contradiction. Hence we many assume that $V(G) - V(C)$ has exactly one component and that here are two vertices $x, y \in V(G) - V(C)$ such that $xy \in E(G)$ and $N_C(x) = \{v^-, v\}$ for a vertex $v \in V(C)$ and $N_C(y) = \emptyset$. But then $G[\{v^{---}, v^{--}, v^-, v, x, y\}]$ is an induced W, a contradiction. ∎

4 CONCLUDING REMARKS

Our results obtained in this paper and in [FRS] may now be summarized as follows: We have examined Theorem A1 in the light of the Metaconjecture for all forbidden pairs RS with $R \cong C$ and S is one of the graphs $P_4, P_5, P_6, Z_1, Z_2, B$ and W. Hence the two cases where $S \cong C_3$ or $S \cong N$ remain. Note that 'C_3−freeness' is not a reasonable choice, since pancyclicity implies the existence of a C_3. For $S \cong N$ observe that all exceptional graphs of Theorem B15 are also CN-free. Moreover, we have constructed a large variety of classes of exceptional graphs that are CN-free, and there is no indication that this might be a 'simple family' (in the terminology of the Metaconjecture). In addition the classes of CDP_7-free graphs, of CHP_7-free graphs, and of CZ_3-free graphs that are not pancyclic are now completely characterized.

Finally, observe that all exceptional graphs have connectivity $\kappa = 2$.

Corollary C1. Let R, S and T be connected graphs $(R, S, T \ncong P_3)$ and G be a 3-connected graph. Then G is RS-free or G is RST-free implies that G is pancyclic, if $R \cong C$ and S is one of the following graphs $P_4, P_5, P_6, Z_1, Z_2, Z_3, B, N$ or W, or ST is one of the pairs of graphs DP_7 or HP_7.

The case $R \cong C$ and $S \cong N$ has been settled by Shephard [Sh].

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