Factor-criticality and matching extension in DCT-graphs

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Abstract

The class of DCT-graphs is a common generalization of the classes of almost claw-free and quasi claw-free graphs. We prove that every even $(2p+1)$-connected DCT-graph G is p-extendable, i.e. every set of p independent edges of G is contained in a perfect matching of G. This result is obtained as a corollary of a stronger result concerning factor-criticality of DCT-graphs.

Keywords: factor-criticality, matching extension, claw, dominated claw toes.

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1 Introduction

In this paper we consider only finite undirected graphs $G = (V(G),\ E(G))$ without loops and multiple edges. For any set $A \subset V(G)$, $\langle A \rangle$ denotes the subgraph of G induced on A, $G - A$ stands for $\langle V(G) - A \rangle$ and $c(G - A)$ (or $c_o(G - A)$) denotes the number of components (odd components) of $G - A$, respectively (we say that a graph is *odd* or *even*

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if it has an odd or even number of vertices). A set $A \subset V(G)$ such that $c(G - A) > 1$ will be called a *cutset*. If $A, B \subset V(G)$, then we denote $N_A(B) = \{x \in A | xy \in E(G)$ for some $y \in B\}$. If $x \in V(G)$, then we simply denote $N(x) = N_{V(G)}(\{x\})$ and we put $N[x] = N(x) \cup \{x\}$. If H is a graph then we say that G is *H-free* if G does not contain an induced subgraph isomorphic to H. If $H \subset G$ is an induced subgraph of G isomorphic to the star $K_{1,r}$ ($r \geq 3$), then the only vertex of degree r in H is called the *center* of H and the vertices of degree 1 in H are called the *toes* of H. In the special case $r = 3$ we say that H is a *claw*. Whenever vertices of a claw (or of an induced $K_{1,r}$) are listed, the center is always the first vertex of the list. For other notation and terminology not defined here we refer e.g. to [3].

Claw-free graphs have been intensively studied during the last decade. Sumner [11] and independently Las Vergnas [5] proved that every even connected claw-free graph has a perfect matching. In accordance with Tutte's 1-factor theorem, we call a set S such that $c_o(G - S) > |S|$ an *antifactor set*. Sumner [12] proved the following theorem.

Theorem 1.1 [12]. Let G be an even connected graph having no perfect matching and let $S \subset V(G)$ be a minimum antifactor set in G. Then every vertex of S is adjacent to vertices of at least three components of $G - S$.

The following extension of the class of claw-free graphs was introduced in [9]. A graph G is *almost claw-free* if the set of claw centers is independent and, for every claw center $x \in V(G)$, $\langle N(x) \rangle$ is 2-dominated (i.e. there are vertices $d_1, d_2 \in N(x)$ such that $yd_1 \in E(G)$ or $yd_2 \in E(G)$ for every $y \in N(x)$). We denote the class of almost claw-free graphs by $\mathcal{ACF}$. It was shown in [9] that every even connected graph $G \in \mathcal{ACF}$ has a perfect matching.

Another extension of the class of claw-free graphs was introduced in [1]. For two nonadjacent vertices a and b of G, let $J(a, b) = \{y \in N(a) \cap N(b) | N[y] \subset N[a] \cup N[b]\}$ (thus, in particular, $J(a, b) = \emptyset$ if a and b are at distance more than 2). The vertices of $J(a, b)$ are called the *dominators* of the pair $\{a, b\}$. A graph G is *quasi claw-free* (denoted $G \in \mathcal{QCF}$) if $J(a, b) \neq \emptyset$ for every pair of vertices a, b at distance 2. It was shown in [1] that

(i) every claw-free graph is quasi claw-free,

(ii) both $\mathcal{ACF} \setminus \mathcal{QCF}$ and $\mathcal{QCF} \setminus \mathcal{ACF}$ are infinite and

(iii) every even connected graph $G \in \mathcal{QCF}$ has a perfect matching.

It is not difficult to observe that also the class $(\mathcal{ACF} \cap \mathcal{QCF}) \setminus \mathcal{CF}$ is infinite. A simple example of a graph $G \in (\mathcal{ACF} \cap \mathcal{QCF}) \setminus \mathcal{CF}$ is in Fig. 1(a) (centers of claws are indicated by double circles).

The class of DCT-graphs, containing all almost claw-free graphs and all quasi claw-free graphs, was first introduced in [2] in the following way. A claw $\langle\{z, a_1, a_2, a_3\}\rangle$ is said to be *dominated* (or *undominated*) if $J(a_1, a_2) \cup J(a_2, a_3) \cup J(a_3, a_1) \neq \emptyset$ (or $= \emptyset$), respectively. The vertices of $J(a_1, a_2) \cup J(a_2, a_3) \cup J(a_3, a_1)$ are called the *dominators* of the claw. We say that a graph G is a *graph with dominated claw toes*, or, briefly, a *DCT-graph* (denoted $G \in \mathcal{DCT}$) [2] if every claw in G is dominated. Clearly, $\mathcal{QCF} \subset \mathcal{DCT}$. It is easy to see that also $\mathcal{ACF} \subset \mathcal{DCT}$. Indeed, let $\langle\{z, a_1, a_2, a_3\}\rangle$ be a claw of an almost claw-free graph G and, without loss of generality, y a neighbor of z adjacent to a_1 and a_2. Since it is adjacent to z, y does not center a claw and thus $N(y) \subset N[a_1] \cup N[a_2]$. Therefore $J(a_1, a_2) \neq \emptyset$ and $G \in \mathcal{DCT}$. It is easy to see that the class $\mathcal{DCT} \setminus (\mathcal{ACF} \cup \mathcal{QCF})$ is infinite. A simple example of a graph $G \in \mathcal{DCT} \setminus (\mathcal{ACF} \cup \mathcal{QCF})$ is shown in Fig. 1(b).

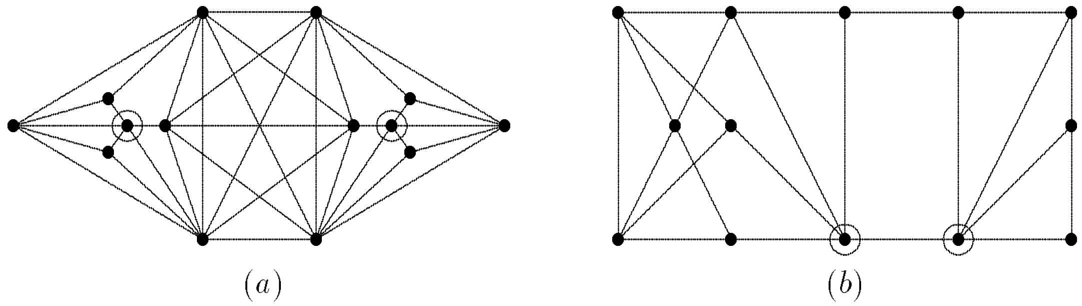

Figure 1

It was proved in [2] that every even connected DCT-graph has a perfect matching.

A graph G of even order n is *p-extendable* [6] if every set of p independent edges is contained in a perfect matching of G. The concept of extendability has been studied in many classes of graphs. In particular, it is known that every $(2p+1)$-connected claw-free graph [7] or almost claw-free graph [10] is p-extendable. A survey on this topic can be found in [8].

In the present paper we generalize these results to the class $\mathcal{DCT}$. The main idea of our proof consists in deleting any p independent edges from G and in showing that the resulting graph has a perfect matching. But actually, when we delete the $2p$ end-vertices of the prescribed edges, we no longer need the information that those vertices induced themselves a graph with a perfect matching. Thus the deletion of any $2p$ vertices leads to the same conclusion. Hence, what we get in our proof is much stronger than the p-extendability and is related to the concept of k-factor-criticality. This property has been defined [4] by an analogy with the concept of factor-critical and bicritical graphs. We say that G is *k-factor-critical* if for every set X of k vertices of G, $G - X$ induces a graph with a perfect matching (or, equivalently, every induced subgraph of order $n - k$ has a perfect matching). With the convention that a graph of order 0 has a perfect matching, it is easy to see that

(i) every graph of order n is n-factor-critical,

(ii) a graph of order n can be k-factor-critical only if k and n are of the same parity,

(iii) any k-factor-critical graph of order n $(2 \leq k < n)$ is $(k-2)$-factor-critical,

(iv) a graph G is 0-factor-critical if and only if G has a perfect matching.

Any $2p$-factor-critical graph is clearly p-extendable.

2 Main result

We first prove the following lemma.

Lemma 2.1. Let G be a graph and $H = \langle\{z, x_1, x_2, \ldots, x_r\}\rangle$ an induced subgraph of G isomorphic to $K_{1,r}$ for some $r \geq 3$. If every subclaw of H is dominated in G, then the set $J = \bigcup_{1\leq i<j\leq r} J(x_i, x_j)$ of the dominators of all the pairs $\{x_i, x_j\}$ satisfies $|J| \geq \frac{r(r-2)}{4}$.

Proof. By the definition of $J(a, b)$, a dominator of a pair $\{x_i, x_j\}$ cannot be adjacent to a third vertex x_h ($h \notin \{i, j\}$) and thus no two different pairs of toes of H can have a common dominator. We construct a graph H' with vertex set $V(H') = \{x_1, x_2, \ldots, x_r\}$ and edge set $E(H') = \{x_ix_j \mid J(x_i, x_j) \neq \emptyset, 1 \leq i < j \leq r\}$. Hence H' has at most $|J|$ edges. Since each subclaw of $H = \langle\{z, x_1, x_2, \ldots, x_r\}\rangle$ is dominated, the complement of H' is triangle-free. By Turán's theorem (see e.g. [3], Chapter 7.3), the maximum number of edges in a triangle-free graph on r vertices is at most $\frac{r^2}{4}$, from which $|J| \geq \binom{r}{2} - \frac{r^2}{4} = \frac{r}{2}\left(\frac{2r-2-r}{2}\right) = \frac{r(r-2)}{4}$. ■

Now we can state the main result of this paper.

Theorem 2.2. Let G be a k-connected DCT-graph of order n. Then:
(i) if $n - k$ is odd and $k \geq 1$, then G is $(k-1)$-factor-critical,
(ii) if $n - k$ is even and $k \geq 2$, then G is $(k-2)$-factor-critical.

Proof. We first observe that the second statement of the theorem is an immediate consequence of the first one. Indeed, if G is k-connected with $n-k$ even and $k \geq 2$, then, setting $k' = k-1$, we get that G is also k'-connected with $n - k'$ odd and thus, by (i), G is $(k'-1)$-factor-critical. Hence it is sufficient to prove (i).

Suppose the statement (i) fails and let X be a set of $k-1$ vertices of G such that $n-k$ is odd and the even subgraph $G' = G - X$ has no perfect matching. Let $S \subset V(G')$ be a minimum antifactor set in G' and put $s = |S|$ (note that, because G is k-connected, $s \geq 1$.) Denote by $C_1, \ldots, C_c$ ($c \geq 3$) the components of $G' - S$. Then, by parity, $c \geq s+2$. By Theorem 1.1, each vertex z of S is adjacent to at least three different components of $G' - S$ and thus centers a claw $\langle\{z, a_{i_1}, a_{i_2}, a_{i_3}\}\rangle$, where $a_{i_j} \in V(C_{i_j})$, $j = 1, 2, 3$. Any dominator of this claw, say $y \in J(a_{i_1}, a_{i_2})$, is adjacent to a_{i_1} and a_{i_2}, but has no neighbor in any other C_ℓ, $\ell \notin \{i_1, i_2\}$. Thus $y \notin \bigcup_{i=1}^{c} V(C_i) \cup S$ and hence $y \in X$.

Let $\widehat{G}$ be the graph obtained from G' by contracting every component C_i to a vertex c_i and by deleting possible multiple edges. We denote $C = \{c_1, c_2, \ldots, c_c\}$. For every subset $A \subset X \cup S$ and for any $i = 1, \ldots, c$ denote $e(c_i, A) = |\{c_ix \in E(\widehat{G}) \mid x \in A\}|$ and put $e(C, A) = \sum_{i=1}^{c} e(c_i, A)$. (Equivalently, $e(c_i, A)$ equals the number of vertices of attachment of the component C_i in A). From above, each claw $\langle\{z, c_{i_1}, c_{i_2}, c_{i_3}\}\rangle$ of $\widehat{G}$ centered at a vertex z of S is dominated by vertices of X and each dominator y of the claw has exactly two neighbors in C. Let $J \subset X$ be the set of all the dominators of all the claws of $\widehat{G}$ centered in S and with toes in C and put $j = |J|$.

Since G is k-connected and C is independent, $e(C, S \cup X) \geq ck$. On the other hand, $e(C, S) \leq sr$, where r is the largest integer ($r \geq 3$) such that there exist vertices z in S and $c_{i_1}, c_{i_2}, \ldots, c_{i_r}$ in C for which $\langle \{z, c_{i_1}, c_{i_2}, \ldots, c_{i_r}\} \rangle$ is isomorphic to $K_{1,r}$. Since every vertex in J is adjacent to only two vertices of C, we have

$$e(C, X) \leq 2j + c(|X| - j) = 2j + c(k - 1 - j).$$

This yields

$$ck \leq e(C, S \cup X) = e(C, S) + e(C, X) \leq sr + 2j + c(k - 1 - j),$$

from which $c(j + 1) \leq sr + 2j$ and thus, since $c \geq s + 2$,

$$sj + s + 2 \leq sr.$$

Hence $j \leq r - 1 - \frac{2}{s}$ and thus, by the integrity of j, $j \leq r - 2$. Lemma 2.1 then implies

$$\frac{r(r-2)}{4} \leq j \leq r - 2.$$

From this we get that either $r = 4$ and $j = 2$, or $r = 3$ and $j = 1$ (note that $j > 0$ implies that $r \neq 2$). From $sj + s + 2 \leq sr$ we then get that in both these cases $s \geq 2$. We consider these two cases separately.

<u>Case 1:</u> $j = 1,\ r = 3,\ s \geq 2,\ c \geq s + 2$.

Let $J = \{y\}$ and assume without loss of generality that $N_C(y) = \{c_1, c_2\}$. Each claw $\langle \{z, c_{i_1}, c_{i_2}, c_{i_3}\} \rangle$ centered in S is dominated by y and thus every vertex $z \in S$ is adjacent to both c_1 and c_2 and, since $r = 3$, to exactly one vertex $c_i \in C \setminus \{c_1, c_2\}$. On the other hand, since G is k-connected, every c_i has at least one neighbor in S. Since $|C \setminus \{c_1, c_2\}| \geq |S|$, this implies that $|N_S(c_i)| = 1$ for every $i, 3 \leq i \leq c$. Let $N_S(c_3) = \{z\}$. Then $(X \setminus \{y\}) \cup \{z\}$ is a cutset of G having $|X| = k - 1$ elements, a contradiction.

<u>Case 2:</u> $j = 2,\ r = 4,\ s \geq 2,\ c \geq s + 2$.

Since $r = 4$, we have $|N_C(J)| = 4$, for otherwise we have an induced $K_{1,4}$ containing an undominated claw. We can assume without loss of generality that $J = \{y_1, y_2\}$ and that $N_C(y_1) = \{c_1, c_2\}$, $N_C(y_2) = \{c_3, c_4\}$ and $N_C(z) = \{c_1, c_2, c_3, c_4\}$ with $z \in S$. Then $y_1y_2 \notin E(G)$ (since otherwise $y_2 \in N[y_1] \setminus (N[c_1] \cup N[c_2])$, contradicting the fact that $y_1 \in J(c_1, c_2)$), and every claw centered in S and with toes in C has $\{c_1, c_2\}$ or $\{c_3, c_4\}$ as a pair of toes.

Suppose first that $c \geq 5$ and put $C' = \{c_5, \ldots, c_c\}$. Every vertex of S has at most one neighbor in C' for otherwise this vertex would center an undominated claw. On the other hand, if there is a $c_i \in C'$ such that $|N_S(c_i)| \leq 2$, then $(X \setminus \{y_1, y_2\}) \cup N_S(c_i)$ is a cutset of G having at most $|X| = k - 1$ elements. Hence $|N_S(c_i)| \geq 3$ for every $c_i \in C'$. This implies $3(c - 4) \leq e(C', S \setminus \{z\}) \leq s - 1$, from which, using $s \leq c - 2$, we get $c \leq \frac{9}{2}$, a contradiction of the assumption $c \geq 5$.

Therefore it remains to consider the case $j = 2, \; r = 4, \; s = 2, \; c = 4$. But then the set $(X \setminus \{y_1, y_2\}) \cup S$ is a cutset of G separating $\langle\{c_1, y_1, c_2\}\rangle$ and $\langle\{c_3, y_2, c_4\}\rangle$ and having $|X| = k - 1$ elements. This contradiction completes the proof. ■

Corollary 2.3. Every even $(2p + 1)$-connected DCT-graph is p-extendable.

Remark. It was also proved in [10] that if G is a $(2p + 1)$-connected $K_{1,p+3}$-free graph such that the set of all claw centers is independent, then G is p-extendable. It can be easily seen that this result and our Corollary 2.3 are independent since the claw centers in a DCT-graph are not necessarily independent and, on the other hand, the claws in a $K_{1,p+3}$-free graph with independent claw centers are not necessarily dominated.

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