On a closure concept in claw-free graphs

Zdeněk Ryjáček
Department of Mathematics
University of West Bohemia
Univerzitní 22
306 14 Plzeň, Czech Republic

September 25, 1996

Abstract

If G is a claw-free graph, then there is a graph $cl(G)$ such that
(i) G is a spanning subgraph of $cl(G)$,
(ii) $cl(G)$ is a line graph of a triangle-free graph, and
(iii) the length of a longest cycle in G and in $cl(G)$ is the same.
A sufficient condition for hamiltonicity in claw-free graphs, the equivalence of some conjectures on hamiltonicity in 2-tough graphs and the hamiltonicity of 7-connected claw-free graphs are obtained as corollaries.

1 Introduction

In this paper, a *graph* will be a finite undirected graph $G = (V(G), E(G))$ without loops and multiple edges. For terminology and notation not defined here we refer to [1]. For any set $A \subset V(G)$ we denote by $\langle A \rangle$ the induced subgraph on A, $G - A$ stands for $\langle V(G) \setminus A \rangle$ and $\omega(G - A)$ denotes the number of components of $G - A$. The (vertex) connectivity of G will be denoted by $\kappa(G)$ and the circumference of G (i.e., the length of a longest cycle in G) will be denoted by $c(G)$. The line graph of a graph G will be denoted by $L(G)$. By a *clique* we mean a (not necessarily maximal) complete subgraph of G.

If H is a graph, then we say that a graph G is *H-free* if G contains no copy of H as an induced subgraph. Specifically, the four-vertex star $K_{1,3}$ will be also called the *claw* and in this case we say that G is *claw-free*. Whenever vertices of a claw are listed, its *center* (i.e., the only vertex of degree 3) will be always the first vertex of the list. It is well known (and can be easily checked) that every line graph is claw-free.

For a vertex $x \in V(G)$, the set $N_G(x) = \{y \in V(G) : \ xy \in E(G)\}$ is called the *neighborhood* of x in G. We say that x is a *locally connected vertex* if $\langle N_G(x) \rangle$ is a connected graph. The set of all locally connected vertices of G will be denoted by $M_{loc}(G)$. If $M_{loc}(G) = V(G)$, then we say that G is *locally connected.*

Oberly and Sumner [7] proved that every connected, locally connected claw-free graph on at least three vertices is hamiltonian. This result was later on strengthened in many directions; from one of these results (see [4]) it e.g. follows that a claw-free graph G is hamiltonian if $M_{loc}(G)$ is a dominating set (i.e., every vertex in $V(G) \setminus M_{loc}(G)$ has a neighbor in $M_{loc}(G)$) and $\langle M_{loc}(G) \rangle$ is connected.

A graph G is said to be t*-tough* (where $t \geq 0$ is a real number) if $|S| \geq t \cdot \omega(G - S)$ for every set $S \subset V(G)$ with $\omega(G - S) > 1$. The *toughness* $\tau(G)$ of G is the maximum value of t for which G is t-tough ($\tau(K_n) = \infty$ for every $n \geq 1$). In [2], Chvátal conjectured the existence of an integer t_0 such that every t_0-tough graph is hamiltonian. Since it is known [3] that for every $\varepsilon > 0$ there is a $(2 - \varepsilon)$-tough graph on at least 3 vertices containing no 2-factor, the smallest such value of t_0 can be 2. The following conjecture is usually attributed to Chvátal.

Conjecture A. *Every 2-tough graph on at least three vertices is hamiltonian.*

It is easy to observe that, for any graph G, $\tau(G) \leq \kappa(G)/2$. Matthews and Sumner [6] proved that if G is claw-free, then $\tau(G) = \kappa(G)/2$. Conjecture A, if true, therefore implies the following conjecture (by Matthews and Sumner).

Conjecture B [6]. *Every 4-connected claw-free graph is hamiltonian.*

Since every line graph is claw-free, the following conjecture by Thomassen [9] is a special case of Conjecture B.

Conjecture C [9]. *Every 4-connected line graph is hamiltonian.*

The following conjecture is a special case of Conjecture C.

Conjecture D. *Every 4-connected line graph of a triangle-free graph is hamiltonian.*

Zhan [10] and independently Jackson [5] proved the analogue of Conjectures C and D in the case of 7-connected line graphs.

Theorem E [10], [5]. *Every 7-connected line graph is hamiltonian.*

In the main result of this paper we show that for every claw-free graph G there is a graph $cl(G)$ such that G is a spanning subgraph of $cl(G)$, $cl(G)$ is a line graph of a triangle-free graph and $c(G) = c(cl(G))$. In Section 3 we prove the following corollary of this result.

Theorem.

(i) Conjectures B, C and D are equivalent.
(ii) Every 7-connected claw-free graph is hamiltonian.

2 Main result

We begin with the following simple lemma.

Lemma 1. *Let G be a graph such that, for every $x \in V(G)$, $\langle N(x)\rangle$ is either a clique or a disjoint union of two cliques. Then there is a triangle-free graph H such that $G = L(H)$.*

Proof. Suppose G satisfies the assumptions of the lemma. We can suppose without loss of generality that G is connected (otherwise we apply the proof to every component of G) and $E(G) \neq \emptyset$ (otherwise apparently $G = L(K_2)$). Let $\mathcal{K} = \{K_1, \ldots, K_\ell\}$ be the collection of maximal cliques in G. By the assumptions of the lemma, $|V(K_i) \cap V(K_j)| \leq 1$ for every $i \neq j$, and $|\{K_i : x \in V(K_i)\}| \leq 2$ for every $x \in V(G)$. Let $X = \{x_1, \ldots, x_k\}$ be the set of vertices of G which lie in exactly one clique of $\mathcal{K}$ and let H be the graph with vertex set $V(H) = X \cup \mathcal{K}$ and with edge set $E(H) = \{K_iK_j : |V(K_i) \cap V(K_j)| = 1, i \neq j\} \cup \{x_mK_j : x_m \in V(K_j)\}$. It is straightforward to check that $G = L(H)$.

Suppose that H contains a triangle T. Then, by the definition of H, $V(T) \subset \mathcal{K}$ (since vertices from X have degree 1 in H). Let $V(T) = \{K_{i_1}, K_{i_2}, K_{i_3}\}$ and let $v \in V(K_{i_1}) \cup V(K_{i_2})$. Then v is locally connected (in G) and thus, by the assumptions of the lemma, $\langle N_G(v)\rangle$ is a clique. This implies $K_{i_1} = K_{i_2}$, a contradiction. ■

The following proposition shows that replacing the neighborhood of a locally connected vertex of a claw-free graph G by a clique affects neither the claw-freeness nor the circumference of G.

Proposition 2. *Let G be a claw-free graph and let x be a locally connected vertex of G such that $\langle N_G(x)\rangle$ is not complete. Let $N' = \{uv : u, v \in N_G(x), uv \notin E(G)\}$ and let G' be the graph with vertex set $V(G') = V(G)$ and with edge set $E(G') = E(G) \cup N'$. Then*
(i) the graph G' is claw-free,
(ii) $c(G') = c(G)$.

Proof. *(i)* Suppose G' is not claw-free and let H be a claw in G'. Since G is claw-free, $|E(H) \cap N'| \geq 1$; since $\langle N_{G'}(x)\rangle$ is a clique, $|E(H) \cap N'| \leq 1$. Denote $H = \langle\{z, y_1, y_2, y_3\}\rangle$, where $zy_1 \in N'$. Then $xy_2 \notin E(G)$ (since otherwise $y_2 \in N_G(x)$ and, by the construction of G', $y_1y_2 \in E(G')$), and similarly $xy_3 \notin E(G)$. But then $\langle\{z, x, y_2, y_3\}\rangle$ is a claw in G, which is a contradiction. Hence G' is claw-free.

(ii) We show that $c(G) = c(G')$. Since obviously $c(G) \leq c(G')$, it is sufficient to prove that for every longest cycle C' in G' there is a cycle C in G such that $V(C') =$

$V(C)$. Let, on the contrary, C' be a longest cycle in G' such that there is no cycle C in G with $V(C') = V(C)$. Then $E(C') \cap N' \neq \emptyset$ (since otherwise C' is a cycle in G) and $N_G(x) \cup \{x\} \subset V(C')$ (since $\langle N_{G'}(x)\rangle$ is complete and C' is a longest cycle in G'). Denote by $P_1, \ldots, P_k$ the components of the graph $(V(C') \setminus \{x\}, E(C' - x) \setminus N')$ and put $\mathcal{P}(C') = \{P_1, \ldots, P_k\}$. Then each P_i is a (trivial or nontrivial) path in G with endvertices in $N_G(x)$. Suppose that the cycle C' is chosen such that, among all cycles in G' with vertex set $V(C')$, $k = |\mathcal{P}(C')|$ is minimum.

Choose one of the orientations of C' and, for any $v \in V(C')$, denote by v^- and v^+ the predecessor and successor of v on C', respectively. For any $v_1, v_2 \in V(C')$, denote by $v_1 C' v_2$ or $v_1 \overleftarrow{C'} v_2$ the consecutive vertices on C' from v_1 to v_2 in the same or opposite orientation with respect to the orientation of C' (if $v_1 = v_2$, then we define both $v_1 C' v_2$ and $v_1 \overleftarrow{C'} v_2$ as a single vertex). Denote by y_i^1, y_i^2 the endvertices of P_i, $i = 1, \ldots, k$ (not excluding the possible case $y_i^1 = y_i^2$) and let the numbering of the paths P_i and of their endvertices be chosen such that $y_1^1 = x^+$, $y_k^2 = x^-$ and $y_{i+1}^1 = (y_i^2)^+, i = 1, \ldots, k-1$.

We show that $y_i^r y_j^s \notin E(G)$ for every $r, s \in \{1, 2\}$ and $i, j \in \{1, \ldots, k\}, i \neq j$. Indeed, if e.g. $y_1^2 y_2^1 \in E(G)$, then, replacing in $\mathcal{P}(C')$ the paths P_1, P_2 by the path $y_1^1 P_1 y_1^2 y_2^1 P_2 y_2^2$, we have a contradiction with the minimality of k. Other cases are similar.

Now, if $k \geq 3$, then $\langle \{x, y_1^1, y_2^1, y_3^1\}\rangle$ is a claw in G and if $k = 1$, then $C = x y_1^1 P_1 y_1^2 x$ is a cycle in G with $V(C) = V(C')$. Hence $k = 2$ and $\mathcal{P}(C') = \{P_1, P_2\}$.

If $y_1^1 \neq y_1^2$ and $y_1^1 y_1^2 \notin E(G)$, then $\langle \{x, y_1^1, y_1^2, y_2^1\}\rangle$ is a claw in G. Hence either $y_1^1 = y_1^2$ or $y_1^1 y_1^2 \in E(G)$ and, by symmetry, $y_2^1 = y_2^2$ or $y_2^1 y_2^2 \in E(G)$.

Since $\langle N_G(x)\rangle$ is connected, there is a path $Q(C')$ in $\langle N_G(x)\rangle$ joining one of y_1^1, y_1^2 to one of y_2^1, y_2^2. Suppose that C' is chosen such that, among all cycles in G' with vertex set $V(C')$ and with $k = |\mathcal{P}(C')| = 2$, $Q(C')$ is shortest possible and assume without loss of generality that $Q(C')$ is a y_1^2, y_2^1-path (otherwise we can modify the cycle C' in $\langle N_{G'}(x)\rangle$ in an obvious way). Let $y_1^2 = x_0, x_1, \ldots, x_\ell = y_2^1$ be the vertices of $Q(C')$. Since $y_1^2 y_2^1 \notin E(G)$, $\ell \geq 2$. Note that, since $\langle N_G(x)\rangle \subset V(C')$, $V(Q(C')) \subset V(C')$.

We now consider $\langle \{x_1, x_1^-, x_1^+, x\}\rangle$. Suppose first that $x_1^- x \in E(G)$ or $x_1^+ x \in E(G)$. If $x_1 \in V(P_2)$, then $|V(P_2)| \geq 3$ (since we already know that $x_1 \notin \{y_2^1, y_2^2\}$) and the cycle $C = x y_1^1 C' y_1^2 x_1 C' y_2^2 y_2^1 C' x_1^- x$ (if $x_1^- x \in E(G)$) or $C = x y_1^1 C' y_1^2 x_1 \overleftarrow{C'} y_2^1 y_2^2 \overleftarrow{C'} x_1^+ x$ (if $x_1^+ x \in E(G)$) is a cycle in G with $V(C) = V(C')$, which is a contradiction. Hence $x_1 \in V(P_1)$, but then similarly $|V(P_1)| \geq 3$ and the cycle $C'' = x x_1^- \overleftarrow{C'} y_1^1 y_1^2 \overleftarrow{C'} x_1 y_2^1 C' y_2^2 x$ (if $x_1^- x \in E(G)$) or $C'' = x x_1^+ C' y_1^2 y_1^1 C' x_1 y_2^1 C' y_2^2 x$ (if $x_1^+ x \in E(G)$) is a cycle in G' with $V(C'') = V(C')$, $|\mathcal{P}(C'')| = 2$ and such that $|V(Q(C''))| = |V(Q(C'))| - 1$, which contradicts the minimality of $Q(C')$. Hence $x_1^- x \notin E(G)$ and $x_1^+ x \notin E(G)$. Finally, if $x_1^- x_1^+ \in E(G)$, then similarly the path P_i containing x_1 has at least 3 vertices and $C'' = x y_1^1 C' x_1^- x_1^+ C' y_1^2 x_1 y_2^1 C' y_2^2 x$ (if $x_1 \in V(P_1)$) or $C'' = x y_1^1 C' y_1^2 x_1 y_2^1 C' x_1^- x_1^+ C' y_2^2 x$ (if $x_1 \in V(P_2)$) is again a cycle in G' with $V(C'') = V(C')$, $|\mathcal{P}(C'')| = 2$ and such that $|V(Q(C''))| = |V(Q(C'))| - 1$. Hence $\langle \{x_1, x_1^-, x_1^+, x\}\rangle$ is a claw. This contradiction proves Proposition 2. ∎

We can now proceed to the definition of the main concept of this paper.

Let G be a claw-free graph. We say that a graph H is a closure *of G, denoted $H = cl(G)$, if*

(i) there is a sequence of graphs $G_1, \ldots, G_t$ such that $G_1 = G$, $G_t = H$, $V(G_{i+1}) = V(G_i)$ and $E(G_{i+1}) = E(G_i) \cup \{uv : \; u, v \in N_{G_i}(x_i), uv \notin E(G_i)\}$ for some $x_i \in V(G_i)$ with connected noncomplete $\langle N_{G_i}(x_i)\rangle$, $i = 1, \ldots, t-1$,

(ii) no vertex of H has a connected noncomplete neighborhood.

(Equivalently, $cl(G)$ is obtained from G by recursively repeating the construction described in Proposition 2, as long as this is possible).

Theorem 3. *Let G be a claw-free graph. Then*

(i) the closure $cl(G)$ is well-defined,

(ii) there is a triangle-free graph H such that $cl(G) = L(H)$,

(iii) $c(G) = c(cl(G))$.

Proof. *(i)* Let H_1, H_2 be two closures of G, suppose that $E(H_1) \setminus E(H_2) \neq \emptyset$ and let $G_1, \ldots, G_t$ be the sequence of graphs that yields H_1. Let j be the smallest integer for which $E(G_j) \setminus E(H_2) \neq \emptyset$ and let $e = uv \in E(G_j) \setminus E(H_2)$. Then, since $e \in E(G_j)$, the vertices u, v have a common neighbor x in G_{j-1} such that $\langle N_{G_{j-1}}(x)\rangle$ is connected. But then, since $E(\langle N_{G_{j-1}}(x)\rangle) \subset E(G_{j-1}) \subset E(H_2)$, $\langle N_{H_2}(x)\rangle$ is connected, hence $e = uv \in E(H_2)$ – a contradiction.

(ii) By part *(i)* of Proposition 2, $cl(G)$ is claw-free and hence, by the construction, the neighborhood of every vertex of $cl(G)$ is either a clique or is a disjoint union of two cliques. By Lemma 1, $cl(G)$ is a line graph of a triangle-free graph.

(iii) $c(G) = c(cl(G))$ immediately by part *(ii)* of Proposition 2. ■

3 Corollaries

Corollary 4. *Let G be a claw-free graph. Then G is hamiltonian if and only if $cl(G)$ is hamiltonian.*

The following corollary shows that Conjectures B, C and D are equivalent.

Corollary 5. *The following statements are equivalent.*

(i) Every 4-connected claw-free graph is hamiltonian.

(ii) Every 4-connected line graph is hamiltonian.

(iii) Every 4-connected line graph of a triangle-free graph is hamiltonian.

Proof. Obviously *(i)* $\Rightarrow$ *(ii)* $\Rightarrow$ *(iii)*. If G is a counterexample to *(i)*, then, by Theorem 3 and by Corollary 4, $cl(G)$ is a counterexample to *(iii)* and hence also *(iii)* $\Rightarrow$ *(i)*. ■

Corollary 6. *Every 7-connected claw-free graph is hamiltonian.*

Proof. If G is a 7-connected nonhamiltonian claw-free graph, then $cl(G)$ is a 7-connected nonhamiltonian line graph, which contradicts Theorem E. ■

Corollary 7. *Let G be a claw-free graph on at least three vertices.*

(i) If $cl(G)$ is a complete graph, then G is hamiltonian.

(ii) [4] If $M_{loc}(G)$ is dominating and $\langle M_{loc}(G)\rangle$ is connected, then G is hamiltonian.

(iii) [7] If G is connected and locally connected, then G is hamiltonian.

Proof. The statement *(i)* follows immediately from Corollary 4; if G satisfies the assumptions of *(ii)* or of *(iii)*, then $cl(G)$ is complete and G is hamiltonian by *(i)*. ■

Example. The graph in Figure 1 is an example of a claw-free graph with a complete closure which satisfies the assumptions of neither part *(ii)* nor part *(iii)* of Corollary 7.

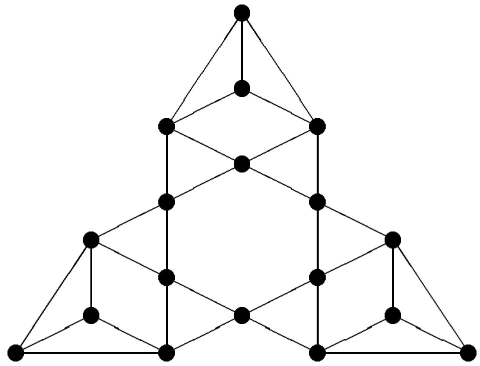

Figure 1: A claw-free graph with a complete closure

Remarks. 1. It is easy to see that $cl(G)$ can be equivalently characterized as the minimum $(K_4 - e)$-free graph on $V(G)$ containing G.

2. If a claw-free graph G is k-connected, or satisfies some of the degree conditions (expressed as a lower bound on $\delta(G)$ or on $\sigma_i(G) = \min\{\sum_{x\in S} d(x) : S \subset V(G)$ independent, $|S| = i\}$ in terms of a function of $n = |V(G)|$), or is N_2-locally connected (see [8]) etc., then apparently so does the closure $cl(G)$. Proofs of many known sufficient conditions for hamiltonicity in claw-free graphs can be therefore simplified by considering $cl(G)$.

3. It would be of interest to observe the behavior of some graph parameters (such as e.g. the independence number) during the process of constructing $cl(G)$. Such observations could possibly yield extensions of some line graph algorithms to the class of claw-free graphs.

Acknowledgement. The main result of this paper was obtained in the stimulating atmosphere of the EIDMA (Euler Institute for Discrete Mathematics and its Applications) workshop on hamiltonicity in 2-tough graphs held in Enschede, The Netherlands, in November 1995. The final shape of the paper was influenced by suggestions made by other participants of the workshop during the lively discussion that followed the presentation of the first version of the proof.

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