2-Factors and Hamiltonicity

Zdeněk Ryjáček
Department of Mathematics
University of West Bohemia
P. O. Box 314
306 14 Plzeň
Czech Republic

Ingo Schiermeyer
Lehrstuhl für Diskrete Mathematik
und Grundlagen der Informatik
TU Cottbus
D-03013 Cottbus
Germany

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Abstract

We prove the following generalization of a result of Faudree and van den Heuvel.

Let G be a 2-connected graph with a 2-factor. If $d(u)+d(v) \geq n-2$ for all pairs of non-adjacent vertices u,v contained in an induced $K_{1,3}$, in an induced $K_{1,3}+e$ or as end-vertices in an induced P_4, then G is hamiltonian.

Keywords: 2-factor, Hamilton cycle, induced subgraph, degree condition.

1 Terminology and notation

We use [2] for terminology and notation not defined here and consider finite simple graphs only.

Let G be a graph on n vertices. We say that G is *hamiltonian* if G has a *Hamilton cycle*, i.e. a cycle containing all vertices of G. If X is a graph, we say that G is *X-free* if G does not contain an induced subgraph isomorphic to X. In this paper we use $K_{1,3}$, $Z_1 \simeq K_{1,3}+e$ and P_4 to denote the graphs of Figure 1. According to the labeling of the vertices we will write $\langle a,b,c,d\rangle \simeq K_{1,3}$, $\langle a,b,c,d\rangle \simeq Z_1$ and $\langle a,b,c,d\rangle \simeq P_4$, respectively.

We will use $\omega(G)$ to denote the number of components of G. A graph G is said to be *t-tough* (cf. [3]) if $t \cdot \omega(G-S) \leq |S|$ for every subset S of $V(G)$ with $\omega(G-S) > 1$. If $v \in V(G)$, then $N(v)$ denotes the set of vertices adjacent to v (the *neighborhood* of v) and $d(v) = |N(v)|$ denotes the degree of v. If we restrict $N(v)$ and $d(v)$ to a subgraph

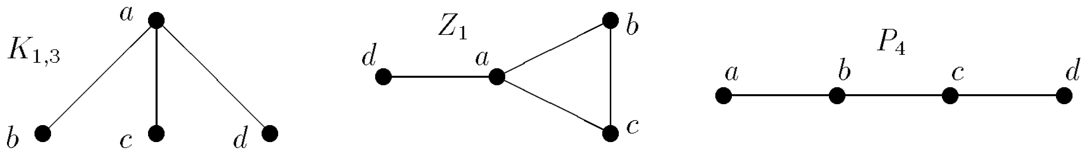

Figure 1.

$F \subset G$, then we will use $N_F(v)$ and $d_F(v)$, respectively. We say that a subgraph $H \subset G$ is a *2-factor* of G if H is a spanning subgraph of G and $d_H(v) = 2$ for every $v \in V(G)$.

Let C be a cycle of G. If an orientation of C is fixed and $u, v \in V(C)$, then by $u \overrightarrow{C} v$ we denote the consecutive vertices on C from u to v in the orientation specified by the orientation of C. The same vertices, in reverse order, are given by $v \overleftarrow{C} u$. If $C \subset G$ is a cycle with a fixed orientation and $v \in V(C)$, then v^+ and v^- denotes the successor and predecessor of v on C, respectively.

2 Main result

Our research was motivated by the following famous conjecture by Chvátal.

Conjecture [3]. *Every 2-tough graph is hamiltonian.*

For the class of 2-tough graphs Enomoto, Jackson, Katerinis and Saito proved the following result.

Theorem 1 [5]. *Every 2-tough graph has a 2-factor.*

Obviously, having a 2-factor is a necessary condition for a graph to be hamiltonian. Moreover, it can be decided in polynomial time whether a given graph G has a 2-factor (see [1]).

The first result for hamiltonicity of graphs having a 2-factor is due to Hoede.

Theorem 2 [7]. *Let G be a connected graph with a 2-factor and let $G_1, \ldots, G_{11}$ be the graphs shown in Fig. 2. If G is $G_1, \ldots, G_{11}$-free, then G is hamiltonian.*

We now turn our attention to degree conditions. The following result by Faudree and van den Heuvel shows that Ore's [8] and Dirac's [4] degree conditions for hamiltonicity can be relaxed under the additional assumption that G has a 2-factor.

Theorem 3 [6]. *Let G be a 2-connected graph with a 2-factor. If $d(u) + d(v) \geq n - 2$ for all pairs of non-adjacent vertices $u, v \in V(G)$, then G is hamiltonian.*

Motivated by Theorem 2, we got the impression that it might be sufficient to require the condition $d(u) + d(v) \geq n - 2$ for all pairs of non-adjacent vertices u, v which are

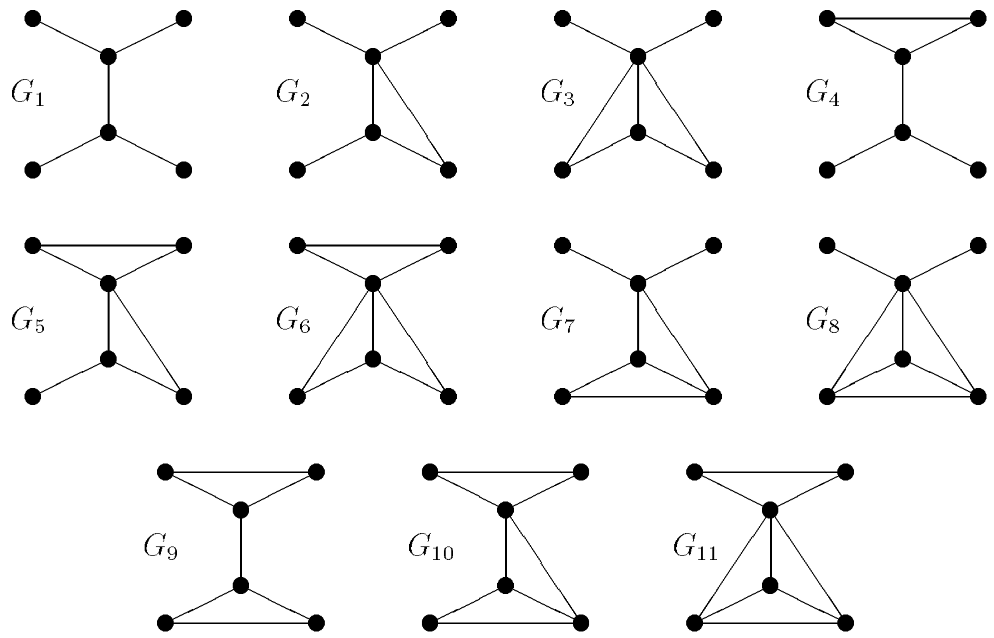

Figure 2.

contained in an induced P_4 or Z_1 (cf. G_1 and G_2 in Fig. 2). However, examples can be given showing that this is not the case even with the requirement $d(u) + d(v) \geq n - 1$. A class of such graphs can be obtained by joining two additional vertices u, v to two prescribed vertices of an arbitrary clique on at least 5 vertices (notice that u and v are contained in an induced $K_{1,3}$ and have $d(u) + d(v) = 4 \leq n - 3$). Thus, the degree condition required for the induced claw is necessary.

Next consider the class of graphs $G_{p,q,r}$ which consist of three complete graphs K_p, K_q, K_r for $p \geq q \geq r \geq 3$ and the additional edges u_iv_i, u_iw_i, v_iw_i for $i = 1, 2$ and vertices $u_1, u_2 \in V(K_p)$, $v_1, v_2 \in V(K_q)$ and $w_1, w_2 \in V(K_r)$. These graphs are 2-connected, claw-free with a 2-factor, but the degree condition is not satisfied for all induced P_4 and induced Z_1.

Finally, the complete bipartite graph $K_{p,q}$ with $p = \lfloor \frac{n-1}{2} \rfloor$ and $q = \lceil \frac{n+1}{2} \rceil$ for $n \geq 5$ is 2-connected, satisfies $d(u) + d(v) \geq n - 2$ for every pair of nonadjacent vertices u, v, but it has no 2-factor.

These examples show that all the assumptions of the following theorem are, in some sense, best possible.

Theorem 4. *Let G be a 2-connected graph with a 2-factor. If $d(u) + d(v) \geq n - 2$ for all pairs of non-adjacent vertices u, v contained in a $K_{1,3}$, in a Z_1 or as endvertices in a P_4, then G is hamiltonian.*

Example. Let i_0, i_1, i_2, i_3, i_4 be integers such that $i_0, i_4 \geq 1$, $i_2 \geq 2$, $i_1 \geq i_2 + i_4 - 1$, $i_3 \geq i_0 + i_2 - 1$. Let G be the graph obtained by taking vertex-disjoint graphs H_0, H_1, H_2, H_3, H_4, where $H_j \simeq K_{i_j}$ for $j = 0, 1, 3, 4$ and $H_2 \simeq \overline{K_{i_2}}$, and by adding all edges xy for $x \in V(H_i)$, $y \in V(H_{i+1})$, $i = 0, 1, 2, 3$. Then the graph G satisfies the assumptions of Theorem 4, but not of Theorem 3. Note that G has diameter $diam(G) = 4$ while the assumptions of Theorem 3 imply $diam(G) \leq 3$.

3 Proofs

We first prove some lemmas which will be useful for the proof of Theorem 4.

Lemma 1. *Let C_p, C_q and C be three vertex-disjoint cycles with $V(C_p) = \{u_1, \ldots, u_p\}$ and $V(C_q) = \{v_1, \ldots, v_q\}$. If $u_p v_q \in E(G)$ and $d_C(u_1) + d_C(v_1) \geq |V(C)| + 1$, then there is a cycle C' such that $V(C') = V(C_p) \cup V(C_q) \cup V(C)$.*

Proof. Since $d_C(u_1) + d_C(v_1) \geq |V(C)| + 1$, there exists a pair of consecutive vertices $w_1, w_2 \in V(C)$ such that $u_1 w_1, v_1 w_2 \in E(G)$ or $u_1 w_2, v_1 w_1 \in E(G)$ and we can easily construct the desired cycle C'. ■

Lemma 2. *Let C_p and C_q be vertex-disjoint cycles with vertices labeled $u_1, \ldots, u_p$ and $v_1, \ldots, v_q$. Suppose $u_p v_q \in E(G)$; $u_p v_1, u_1 v_q, u_1 v_1 \notin E(G)$. If $d_{C_p \cup C_q}(u_1) + d_{C_p \cup C_q}(v_1) \geq p + q - 1$, then there is a cycle C such that $V(C) = V(C_p) \cup V(C_q)$.*

Proof. Suppose there is no such cycle. Then $v_1 u_{p-1}, v_{q-1} u_1 \notin E(G)$. Let

$$S = \{i \mid v_1 u_i \in E(G), 2 \leq i \leq p-2\},\ T = \{i \mid u_1 u_{i+1} \in E(G), 1 \leq i \leq p-2\}.$$

If there is some $i \in T \cap S$, then $C = v_1 u_i \overleftarrow{C_p}\, u_1 u_{i+1} \overrightarrow{C_p}\, u_p v_q \overleftarrow{C_q}\, v_1$ would be the desired cycle. Hence we can assume that $S \cap T = \emptyset$. Now $d_{C_p}(v_1) = |S|$ and $d_{C_p}(u_1) = |T| + 1$, from which $d_{C_p}(u_1) + d_{C_p}(v_1) = |S| + |T| + 1 = |S \cup T| + 1 \leq p - 1$. By the same argument we obtain $d_{C_q}(u_1) + d_{C_q}(v_1) \leq q - 1$ and thus $d_{C_p \cup C_q}(u_1) + d_{C_p \cup C_q}(v_1) \leq p + q - 2$, a contradiction. ■

Let C^1, C^2 be two vertex-disjoint cycles. We say that a vertex $v \in V(C^1)$ is C^2-*universal*, if v is adjacent to all vertices of C^2.

Assume now that there are two vertex-disjoint cycles C^1, C^2 and a C^2-universal vertex $v \in V(C^1)$. If v^- or v^+ has a neighbor on C^2, then we can again easily construct a cycle C such that $V(C) = V(C^1) \cup V(C^2)$.

Lemma 3. *Let G be a non-hamiltonian graph with a 2-factor consisting of $k \geq 2$ cycles C^1, $C^2, \ldots,$ C^k, where k is minimal. Then for every pair of cycles C^i, C^j, $1 \leq i < j \leq k$, and every C^j-universal vertex $v \in V(C^i)$, neither v^- nor v^+ has a neighbor on C^j.*

Corollary 4. *Let G be a non-hamiltonian graph with a 2-factor consisting of $k \geq 2$ cycles $C^1,\ C^2, \ldots,\ C^k$, where k is minimal. Then for every pair of cycles $C^i,\ C^j,\ 1 \leq i < j \leq k$, all C^j-universal vertices of $V(C^i)$ are pairwise non-consecutive.*

Corollary 5. *Let G be a non-hamiltonian graph with a 2-factor consisting of $k \geq 2$ cycles $C^1,\ C^2, \ldots,\ C^k$, where k is minimal. Then there is no pair of cycles $C^i,\ C^j,\ 1 \leq i < j \leq k$, such that there is both a C^j-universal vertex $v_i \in V(C^i)$ and a C^i-universal vertex $v_j \in V(C^j)$.*

We will also use the following simple lemma.

Lemma 6. *Let C be a cycle in a graph G and let $x,\ y \in V(C)$ be such that there is no x,y-path P with $V(P) = V(C)$. Then $x^+y^+ \notin E(G)$ and $d_C(x^+) + d_C(y^+) \leq |V(C)|$.*

Proof. If $x^+y^+ \in E(G)$, then $P = x \overleftarrow{C}\, y^+x^+ \overrightarrow{C}\, y$ is a x,y-path with $V(P) = V(C)$. Hence $x^+y^+ \notin E(G)$. Put $M = \{z \in V(C) |\ zx^+ \in E(G)\}$ and $N = \{z \in x^{++} \overrightarrow{C}\, y^+ |\ z^-y^+ \in E(G)\} \cup \{z \in y^{++} \overrightarrow{C}\, x |\ z^+y^+ \in E(G)\}$. Then $|M| = d_C(x^+)$, $|N| = d_C(y^+) - 1$ and $x^+ \notin M \cup N$. Thus, if $d_C(x^+) + d_C(y^+) \geq |V(C)| + 1$, there is a vertex $z \in M \cap N$, but then the path $x \overleftarrow{C}\, y^+z^- \overleftarrow{C}\, x^+z \overrightarrow{C}\, y$ (if $z \in x^+ \overrightarrow{C}\, y^+$) or $x \overleftarrow{C}\, z^+y^+ \overrightarrow{C}\, zx^+ \overrightarrow{C}\, y$ (if $z \in y^+ \overrightarrow{C}\, x^+$) yields a contradiction. Hence $d_C(x^+) + d_C(y^+) \leq |V(C)|$. ■

Proof of Theorem 4. Assume G is not hamiltonian and choose a 2-factor of G with $k \geq 2$ cycles $C^1,\ C^2, \ldots, C^k$ such that k is minimal. We distinguish the following cases.

Case 1. There are two cycles $C^{t_1},\ C^{t_2},\ 1 \leq t_1 < t_2 \leq k$, which are connected by two vertex-disjoint edges.

Subcase A. There is an edge xy such that $x \in V(C^{t_1}),\ y \in V(C^{t_2})$ and neither x is C^{t_2}-universal nor y is C^{t_1}-universal.

Subcase B. Every vertex $x \in V(C^{t_1})$ with $N(x) \cap V(C^{t_2}) \neq \emptyset$ is C^{t_2}-universal.

Case 2. No pair of cycles $C^i,\ C^j,\ 1 \leq i < j \leq k$, is connected by two vertex-disjoint edges.

By Corollary 5, no other possibilities can occur.

Throughout the proof, we denote $n_i = |V(C^i)|,\ 1 \leq i \leq k$. For convenience we set $p = n_1$ and $q = n_2$.

Case 1. We can without loss of generality suppose that $C^{t_1} = C^1 \simeq C_p$ with vertices labeled $u_1,\ \ldots, u_p$, $C^{t_2} = C^2 \simeq C_q$ with vertices labeled $v_1,\ \ldots, v_q$, $u_pv_q \in E(G)$ and $u_iv_j \in E(G)$ for some i, j with $1 \leq i \leq p-1$, $1 \leq j \leq q-1$.

Subcase A. Suppose (without loss of generality) that $u_pv_1, u_1v_q, u_1v_1 \notin E(G)$. Thus $\langle u_1, u_p, v_q, v_1\rangle \simeq P_4$, from which $d(u_1)+d(v_1) \geq n-2$. Since k is minimal, by Lemma 1 and Lemma 2 we have $d_{C^1}(u_1)+d_{C^1}(v_1) = p-1,\ d_{C^2}(u_1)+d_{C^2}(v_1) = q-1$. If $u_1u_{i+1}, v_1v_{j+1} \in$

$E(G)$, then the cycle $u_1u_{i+1} \overrightarrow{C^1} u_pv_q \overleftarrow{C^2} v_{j+1}v_1 \overrightarrow{C^2} v_ju_i \overleftarrow{C^1} u_1$ contradicts the minimality of k. Hence we can without loss of generality assume that $u_1u_{i+1} \notin E(G)$. Since equality holds in Lemma 2, this implies $v_1u_i \in E(G)$ and thus $2 \leq i \leq p-2$. Moreover, since $v_1u_{p-1} \notin E(G)$, there exists $r > i$ such that $u_{r-1}v_1, u_{r+1}u_1 \in E(G)$ and $u_1u_r, v_1u_r \notin E(G)$. Since there is no cycle C such that $V(C) = V(C^1) \cup V(C^2)$, we have $u_rv_2, u_rv_q \notin E(G)$. By symmetry and since $u_rv_1 \notin E(G)$, we conclude $u_rv_{q-1} \notin E(G)$. Now $C = v_1u_{r-1} \overleftarrow{C^1} u_1u_{r+1} \overrightarrow{C^1} u_pv_q \overleftarrow{C^2} v_1$ is a cycle such that $V(C) = V(C^1) \cup V(C^2) \setminus \{u_r\}$. If $u_ru_i, u_ru_{i+1} \in E(G)$ for some i with $2 \leq i \leq r-2$ or $r+1 \leq i \leq p-1$, then u_r can be *inserted* into the cycle C by replacing the edge u_iu_{i+1} by the path $u_iu_ru_{i+1}$. Hence we conclude that $u_ru_{r-2}, u_ru_{r+2} \notin E(G)$ and $d_{C^1}(u_r) \leq p/2$. Likewise u_r can be inserted if $u_rv_i, u_rv_{i+1} \in E(G)$ for some i with $2 \leq i \leq q-3$. Hence $d_{C^2}(u_r) \leq (q-4+1)/2 = (q-3)/2$. For any other cycle C^j, $3 \leq j \leq k$, if $u_rw_1, u_rw_2 \in E(G)$ for two consecutive vertices w_1, w_2 on C^j, then u_r can be inserted into C^j, contradicting the minimality of k. Hence $d_{C^j}(u_r) \leq n_j/2$ and thus $d(u_r) \leq p/2 + (q-3)/2 + \sum_{j=3}^{k} n_j/2 = (n-3)/2$. Now $\langle u_{r-1}, u_{r-2}, u_r, v_1\rangle$ and $\langle u_{r+1}, u_r, u_{r+2}, u_1\rangle$ are isomorphic to $K_{1,3}$ or Z_1 implying $d(v_1) \geq (n-1)/2$ and $d(u_1) \geq (n-1)/2$. Altogether we obtain $n-1 \leq d(u_1)+d(v_1) \leq p+q-2+\sum_{j=3}^{k} n_j = n-2$, a contradiction.

Subcase B. Let $M = \{x \in V(C^1) |\ N_{C^2}(x) \neq \emptyset\}$. Then, by the assumptions of Case 1, $|M| \geq 2$, $u_p \in M$ and (recall Corollary 5 and Corollary 4), no two vertices in M are consecutive on C^1. Suppose first that there are $x, y \in M$, $x \neq y$, such that both $x^-x^+ \notin E(G)$ and $y^-y^+ \notin E(G)$. Then, since (by Lemma 3) both $\langle x, x^-, x^+, v_q\rangle \simeq K_{1,3}$ and $\langle y, y^-, y^+, v_q\rangle \simeq K_{1,3}$, we have $d(x^-)+d(x^+)+d(y^-)+d(y^+) \geq 2(n-2) \geq 2(p+q-2+n-p-q) \geq 2(p+1)+2(n-p-q)$. On the other hand, by the minimality of k, there is no hamiltonian x,y-path in $G[V(C^1)]$ and hence, by Lemma 6, $d_{C^1}(x^+)+d_{C^1}(y^+)+d_{C^1}(x^-)+d_{C^1}(y^-) \leq 2p$. Together we obtain $2(p+1)+2(n-p-q) \leq d(x^+)+d(y^+)+d(x^-)+d(y^-) \leq 2p+2(n-p-q)$, which is a contradiction.

Hence we can suppose that $x^-x^+ \in E(G)$ for every $x \in M$, $x \neq u_p$. But then, for any $x \in M$, $x \neq u_p$, we have $u_1x \notin E(G)$ and $u_1x^{++} \notin E(G)$ (otherwise the cycles $u_1xv_1 \overrightarrow{C^2} v_qu_p \overleftarrow{C^1} x^+x^- \overleftarrow{C^1} u_1$ and $u_1x^{++} \overrightarrow{C^1} u_pv_q \overleftarrow{C^2} v_1xx^+x^- \overleftarrow{C^1} u_1$ contradict the minimality of k). Now $x^{++} \notin M$, since $x^{++} \overrightarrow{C^1} x^-x^+x$ is a hamiltonian path in $G[V(C^1)]$. Since also (by Lemma 6) $u_1x^+ \notin E(G)$ and, by Lemma 3, $d_{C^2}(u_1) = 0$, we have $d_{C^1\cup C^2}(u_1) \leq p-1-3(|M|-1)$. Since every vertex in M is C^2-universal, we have $d_{C^1\cup C^2}(v_q) \leq q-1+|M|$. If there is a cycle C^i, $3 \leq i \leq k$, such that u_1 and v_q have consecutive neighbors on C^i, then we easily construct a cycle C' with $V(C') = V(C^1)\cup V(C^2)\cup V(C^i)$, contradicting the minimality of k; hence $d_{C^3\cup\ldots\cup C^k}(u_1) + d_{C^3\cup\ldots\cup C^k}(v_q) \leq |V(C^3) \cup \ldots \cup V(C^k)| = n-p-q$. Since $\langle u_p, v_q, v_1, u_1\rangle \simeq Z_1$, we have $d(u_1) + d(v_q) \geq n-2$. Altogether we obtain $n-2 \leq d(u_1) + d(v_q) \leq p-1-3(|M|-1)+q-1+|M|+n-p-q$, from which $|M| \leq 3/2$, a contradiction.

Case 2. Since G is 2-connected, there are m cycles, $3 \leq m \leq k$, say, $C^1, C^2, \ldots, C^m$, with vertices labeled $v_1^i, \ldots, v_{n_i}^i$, and pairs of vertices $v_{r_i}^i, v_{s_i}^i \in V(C^i)$ such that $v_{s_i}^i v_{r_{i+1}}^{i+1} \in E(G)$ (modulo m). If $s_i = r_i \pm 1$ for all $1 \leq i \leq m$, then there is a cycle C such that

$V(C) = \cup_{i=1}^m V(C^i)$, e.g. $C = v_{s_1}^1 v_{r_2}^2 \overrightarrow{C^2} v_{s_2}^2 v_{r_3}^3 \ldots v_{s_m}^m v_{r_1}^1 \overrightarrow{C^1} v_{s_1}^1$, if $s_i = r_i + 1$ for $1 \leq i \leq m$, which contradicts the minimality of k.

Now suppose without loss of generality that $s_1 \neq r_1 \pm 1$. Thus $n_1 \geq 4$. If $v_{r_1+1}^1 v_{s_1+1}^1 \in E(G)$ or $d_{C^1}(v_{r_1+1}^1) + d_{C^1}(v_{s_1+1}^1) \geq n_1 + 1$, then, by Lemma 6, there is a hamiltonian path in $G[V(C^1)]$ with endvertices $v_{r_1}^1$, $v_{s_1}^1$.

Suppose such a path does not exist. With a repeat of previous arguments we will show that $v_{s_1}^1, v_{r_1}^1$ are both universal vertices and that $n_1 = 4$. Suppose first that $v_{s_1}^1$ is not universal. Then there is a vertex $x \in V(C^2)$ such that $v_{s_1}^1 x \in E(G)$, but $v_{s_1}^1 x^+ \notin E(G)$. As in Subcase A we obtain this time $d(v_{s_1+1}^1) + d(x^+) \leq (n_1 - 2) + (n_2 - 1) + \sum_{j=3}^k n_j < n - 2$, a contradiction. The same argument holds for $v_{r_1}^1$. Thus both $v_{s_1}^1$ and $v_{r_1}^1$ are universal vertices. Suppose next that $n_1 \geq 5$. By Lemma 6 we have $d_{C^1}(v_{s_1+1}^1) + d_{C^1}(v_{r_1+1}^1) \leq n_1$. Hence we may assume that $d_{C^1}(v_{s_1+1}^1) \leq n_1/2$. But then $\langle v_{s_1}^1, x, x^+, v_{s_1+1}^1 \rangle \simeq Z_1$ for any pair of consecutive vertices $x, x^+ \in V(C^2)$ and $d_{C^1 \cup C^2}(v_{s_1+1}^1) + d_{C^1 \cup C^2}(x^+) \leq n_1/2 + (n_2 - 1 + 1) + \sum_{j=3}^k n_j < (n_1 - 2) + n_2 + \sum_{j=3}^k n_j \leq n - 2$, a contradiction. Hence $n_1 = 4$.

Let $\{s_1, r_1\} = \{2, 4\}$. Then $d_{C^1}(v_1^1) = d_{C^1}(v_3^1) = 2$ and both v_2^1 and v_4^1 are contained in an induced Z_1, say, $\langle v_2^1, v_1^1, v_{n_m}^m, v_1^m \rangle$ and $\langle v_4^1, v_3^1, v_{n_2}^2, v_1^2 \rangle$. Since $N_{C^m}(v_3^1) = \emptyset$, $N_{C^m}(v_1^1) = \emptyset$, $N_{C^2}(v_3^1) = \emptyset$, $N_{C^2}(v_1^1) = \emptyset$, we have $d_{C^1 \cup C^2 \cup C^3}(v_1^1) + d_{C^1 \cup C^2 \cup C^3}(v_3^1) = 4$, where $n_1 + n_2 + n_3 \geq 4 + 3 + 3 = 10$. Since $d(v_1^1) + d(v_3^1) \geq n - 2$, we have $k \geq 4$ and $\sum_{j=4}^k d_{C^i}(v_1^1) + d_{C^i}(v_3^1) \geq \sum_{j=4}^k n_j + 4$. Hence there exists a cycle C^j and two consecutive vertices w_1, w_2 on C^j such that (without loss of generality) $v_1^1 w_1, v_3^1 w_2 \in E(G)$. Then $C^a = v_4^1 v_1^2 \overrightarrow{C^2} v_{n_2}^2 v_4^1$ and $C^b = v_1^1 v_2^1 v_3^1 w_2 \overrightarrow{C^j} w_1 v_1^1$ are two cycles such that $V(C^a) \cup V(C^b) = V(C^1) \cup V(C^2) \cup V(C^j)$, which contradicts the minimality of k.

This shows that, for each cycle C^i, the vertices $v_{r_i}^i$ and $v_{s_i}^i$ are connected by a hamiltonian path in $G[V(C^i)]$, $1 \leq i \leq m$. But then there is a cycle C such that $V(C) = \cup_{j=1}^m V(C^j)$, contradicting again the minimality of k. This contradiction completes the proof of Theorem 4. ∎

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