Closure and Hamiltonian-Connectivity of Claw-Free Graphs

Béla Bollobás
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
U.S.A.
e-mail: bollobas@msci.memphis.edu

Oliver Riordan
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
U.S.A.
e-mail: omr10@cam.ac.uk

Zdeněk Ryjáček
Department of Mathematics
University of West Bohemia
Univerzitní 22, 306 14 Plzeň
Czech Republic
e-mail: ryjacek@kma.zcu.cz

Akira Saito
Department of Mathematics
Nihon University
Sakurajosui 3–25–40
Setagaya-ku, Tokyo 156
JAPAN
e-mail: asaito@math.chs.nihon-u.ac.jp

R.H. Schelp[1]
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
U.S.A.
e-mail: schelpr@mathsci.msci.memphis.edu

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Abstract

In [3], the closure $cl(G)$ for a claw-free graph G is defined, and it is proved that G is hamiltonian if and only if $cl(G)$ is hamiltonian. On the other hand, there exist infinitely many claw-free graphs G such that G is not hamiltonian-connected (resp. homogeneously traceable) while $cl(G)$ is hamiltonian-connected (resp. homogeneously traceable). In this paper we define a new closure $cl_k(G)$ ($k \geq 1$) as a generalization of $cl(G)$ and prove the following theorems. (1) A claw-free graph G is hamiltonian-connected if and only if $cl_3(G)$ is hamiltonian-connected. (2) A claw-free graph G is homogeneously traceable if and only if $cl_2(G)$ is homogeneously traceable. We also discuss the uniqueness of the closure.

1. Introduction.

For graph theoretic notation not defined in this paper, we refer the reader to [2]. A vertex x of a graph G is said to be *locally connected* if the neighborhood $N_G(x)$ of x in G induces a connected graph. A locally connected vertex x is said to be *eligible* if $N_G(x)$ induces a noncomplete graph. For a vertex x of a graph G, we consider the operation of joining every pair of nonadjacent vertices in $N_G(x)$ by an edge so that $N_G(x)$ induces a complete graph in the resulting graph. This operation is called *local completion* of G at x. We shall consider a series of local completions at eligible vertices. For a graph G, let $G_0 = G$. For $i \geq 0$, if G_i is defined and it has an eligible vertex x_i, then apply local completion of G_i at x_i to obtain a new graph G_{i+1}. If G_i has no eligible vertex, let $cl(G) = G_i$ and call it the *closure* of G. The above operation was introduced and the following theorems were proved in [3].

Theorem A ([3]). *If G is a claw-free graph, then*

(1) *a graph obtained from G by local completion is also claw-free, and*

(2) *$cl(G)$ is uniquely determined.* □

Theorem B ([3]). *Let G be a claw-free graph. Then G is hamiltonian if and only if $cl(G)$ is hamiltonian.* □

Recently, several other properties on paths and cycles of a claw-free graph and those of its closure were studied by Brandt et al. A graph G is said to be *hamiltonian-connected* if every pair of distinct vertices of G can be joined by a hamiltonian path of G. And G is said to be *homogeneously traceable* if every vertex of G is an endvertex of some hamiltonian path of G. The following theorem was proved in [1].

Theorem C ([1]).

(1) *A claw-free graph G is traceable if and only if $cl(G)$ is traceable.*

(2) *There exist infinitely many claw-free graphs G such that $cl(G)$ is hamiltonian-connected while G is not hamiltonian-connected.*

(3) *There exist infinitely many claw-free graphs G such that $cl(G)$ is homogeneously traceable while G is not homogeneously traceable.* □

However, if we impose some restrictions to the vertices used for local completion, homogeneous traceability and hamiltonian-connectivity may be preserved under closure. This is the motivation of this paper.

A vertex x of a graph G is said to be locally k-connected if $N_G(x)$ induces a k-connected graph. We modify the closure so that we allow local completions only at locally k-connected vertices. More precisely, consider a sequence of local completions $G = G_0, G_1, \ldots, G_r = H$, where G_{i+1} is obtained from G_i by local completion at a locally k-connected vertex for each i, $0 \leq i \leq r-1$. If H does not have an eligible locally k-connected vertex, we call H a *k-closure* of G and denote it by $cl_k(G)$. We prove the following theorem.

Theorem 1. *Let G be a claw-free graph. Then*

(1) *$cl_k(G)$ is uniquely determined for each k,*

(2) *G is hamiltonian-connected if and only if $cl_3(G)$ is hamiltonian-connected, and*

(3) *G is homogeneously traceable if and only if $cl_2(G)$ is homogeneously traceable.*

We first prove (2) in Section 2. The we prove (3) in Section 3. We postpone the proof of (1) until Section 4, where we discuss the uniqueness of the closure in a more generalized situation.

Before closing this section we introduce some notation which is used in the subsequent arguments. For a graph G and $S \subset V(G)$, the subgraph induced by S is denoted by $G[S]$. When we consider a path or a cycle, we always assign an orientation to it. Let $P = x_0 x_1 \cdots x_m$. We call x_0 and x_m the starting vertex and the terminal vertex of P, respectively. We define $x_i^{+(P)} = x_{i+1}$ and $x_i^{-(P)} = x_{i-1}$. Furthermore, we define $x_i^{++(P)} = x_{i+2}$. When it is obvious which path is considered in the context, we sometimes write x_i^+ and x_i^- instead of $x_i^{+(P)}$ and $x_i^{-(P)}$, respectively. For x_i, $x_j \in V(P)$ with $i \leq j$, we denote the subpath $x_i x_{i+1} \cdots x_j$ by $x_i \overrightarrow{P} x_j$. The same path traversed in the opposite direction is denoted by $x_j \overleftarrow{P} x_i$. We also use the same notation for a cycle. A path having x and y as the starting and the terminal vertices, respectively, is called an xy-path.

Given two graphs G and H, we denote by $G \cup H$ the graph with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$.

2. Hamiltonian Connectivity.

A graph G is said to be l-path-connected if for every pair of distinct vertices x and $y \in V(G)$ there exists an xy-path of length at least l in G. Thus, a graph of order n is hamiltonian-connected if and only if it is $(n-1)$-path-connected. In this section, we prove a theorem on l-path-connected graphs, which is stronger than Theorem 1 (2).

Theorem 2. *Let G be a claw-free graph and let l be a positive integer. Then G is l-path-connected if and only if $cl_3(G)$ is l-path-connected.*

The above theorem follows immediately from the following theorem.

Theorem 3. *Let G be a claw-free graph and let x, a, $b \in V(G)$. Suppose $a \neq b$ and $N_G(x)$ induces a 3-connected graph in G. Let G' be the graph obtained from G by local completion at x. If G' has an ab-path of length l, then G has an ab-path of length at least l.*

We prove several lemmas before we prove the above theorem. The first one is an easy observation.

Lemma 4. *Let G be a claw-free graph, and let $x \in V(G)$. Then every induced path in $G[N_G(x)]$ has length at most three.*

Proof. If $G[N_G(x)]$ has an induced path $P = u_0u_1 \dots u_l$ with $l \geq 4$, then $\{x, u_0, u_2, u_4\}$ forms a claw in G. This is a contradiction. $\square$

Lemma 5. *Let G be a claw-free graph and let x, a, $b \in V(G)$ with $a \neq b$. Let G' be the graph obtained from G by local completion at x. If P is a longest ab-path in G' with $E(P) \cap (E(G') - E(G)) \neq \emptyset$, then $N_G(x) \cup \{x\} \subset V(P)$.*

Proof. Let $N = E(G') - E(G)$. Assume $x \notin V(P)$. Since $E(P) \cap N \neq \emptyset$, $uv \in E(P) \cap N$ for some u, $v \in V(G)$. We may assume $u = v^-$. Then $a\overrightarrow{P}uxv\overrightarrow{P}b$ is an ab-path in G', which is longer than P. This is a contradiction. Therefore, $x \in V(P)$.

Assume $N_G(x) - V(P) \neq \emptyset$, say $v \in N_G(x) - V(P)$. If $a \neq x$, then x^- exists and $a\overrightarrow{P}x^-vx\overrightarrow{P}b$ is an ab-path in G', which is longer than P. If $a = x$, then $a^+ \in N_{G'}(x)$ exists and $ava^+\overrightarrow{P}b$ is an ab-path in G', which is longer than P. Therefore, we have a contradiction in either case. $\square$

Lemma 6. *Let G be a claw-free graph and let x, a, $b \in V(G)$ with $a \neq b$. Let G' be the graph obtained from G by local completion at x. Then there exists a longest ab-path P in G' with $|E(P) \cap (E(G') - E(G))| \leq 1$.*

Proof. Let $N = E(G') - E(G)$, and choose a longest ab-path in G' so that $|E(P) \cap N|$ is as small as possible. Assume $|E(P) \cap N| \geq 2$, say $e = u_1v_1$, $f = u_2v_2 \in E(P) \cap N$, $e \neq f$. By Lemma 5, $\{x\} \cup N_G(x) \subset V(P)$. We may assume $u_1 = v_1^-$, $u_2 = v_2^-$ and $v_1 \in a\overrightarrow{P}u_2$. Then $v_1 \neq v_2$ and $u_1 \neq u_2$ (possibly $v_1 = u_2$). Furthermore, $\{u_1, v_1\} \cup \{u_2, v_2\} \subset N_G(x)$ and $\{u_1, v_1\} \cup \{u_2, v_2\}$ induces a complete graph in G'.

If $v_1v_2 \in E(G)$, let $P' = a\overrightarrow{P}u_1u_2\overleftarrow{P}v_1v_2\overrightarrow{P}b$. Then P' is a longest ab-path in G' and $E(P') = E(P) \cup \{u_1u_2, v_1v_2\} - \{e, f\}$. Since $\{e, f\} \subset N$ and $v_1v_2 \notin N$, $|E(P') \cap N| < |E(P) \cap N|$. This is a contradiction. Hence $v_1v_2 \notin E(G)$. Similarly, $u_1u_2 \notin E(G)$. If $x \neq b$, then x^+ exists and x^+, v_1, v_2 are distinct neighbors of x. Since G is claw-free and $v_1v_2 \notin E(G)$, we either have $x^+v_1 \in E(G)$ or $x^+v_2 \in E(G)$. We may assume $x^+v_1 \in E(G)$. If $x^+ \in a\overrightarrow{P}u_1$, then let $P' = a\overrightarrow{P}xu_1\overleftarrow{P}x^+v_1\overrightarrow{P}b$. If $x^+ \in v_1^+\overrightarrow{P}b$, let $P' = a\overrightarrow{P}u_1x\overleftarrow{P}v_1x^+\overrightarrow{P}b$. Then in either case P' is a longest ab-path and $E(P') = (E(P) - \{e, xx^+\}) \cup \{u_1x, v_1x^+\}$. Since $\{u_1x, v_1x^+\} \cap N = \emptyset$ and $e \in N$, we have $|E(P') \cap N| < |E(P) \cap N|$. This is a contradiction.

If $x = b$, we can apply the same arguments as above to $\{b^-, u_1, u_2\} \subset N_G(b)$ and obtain a contradiction. Therefore, the lemma is proved. $\square$

The proof of Theorem 3 is divided into two cases, which deal with $x \notin \{a, b\}$ and $x \in \{a, b\}$, respectively. We present both cases as lemmas. Lemma 7 deals with the first case, while Lemma 8 handles the second case.

Lemma 7. *Let G be a claw-free graph and let x, a, b be distinct vertices in G. Suppose $N_G(x)$ induces a 3-connected graph in G. Let G' be the graph obtained from G by local completion at x. If G' has an ab-path of length l, then G has an ab-path of length at least l.*

Proof. Let m be the length of a longest ab-path in G'. Then $m \geq l$. We prove that G has an ab-path of length at least m. Assume, to the contrary, G has no ab-path of length at least m. Let $N = E(G') - E(G)$. By the definition of local completion, if $uv \in N$, then u, $v \in N_G(x)$. Let $\mathcal{P}$ be the set of all the longest ab-paths in G'. If $E(P) \cap N = \emptyset$ for some $P \in \mathcal{P}$, then P is an ab-path of length m in G. This contradicts the assumption. Therefore, $E(P) \cap N \neq \emptyset$ for each $P \in \mathcal{P}$. Let $\mathcal{P}_0 = \{P \in \mathcal{P} : |E(P) \cap N| = 1\}$. By Lemma 6, $\mathcal{P}_0 \neq \emptyset$.

Let $P \in \mathcal{P}_0$. Then $\{x\} \cup N_G(x) \subset V(P)$ by Lemma 5. Let $E(P) \cap N = \{uv\}$ with $u = v^-$. We may also assume $x \in a^+\overrightarrow{P}u^-$ (possibly $x = u^-$). If $x^-x^+ \in E(G)$, then $a\overrightarrow{P}x^-x^+\overrightarrow{P}uxv\overrightarrow{P}b$ is an ab-path of length m in G. This contradicts the assumption. Thus, $x^-x^+ \notin E(G)$. If $ux^- \in E(G)$, then $a\overrightarrow{P}x^-u\overleftarrow{P}xv\overrightarrow{P}b$ is an ab-path of length m in G, again a contradiction. Hence $ux^- \notin E(G)$. If $vx^+ \in E(G)$, then $a\overrightarrow{P}xu\overleftarrow{P}x^+v\overrightarrow{P}b$ is an ab-path of length m in G. This is a contradiction, and hence $vx^+ \notin E(G)$. If $u \neq x^+$, then $\{x, v, u, x^+\}$ does not form a claw in G, and hence $ux^+ \in E(G)$. By the assumption $v \neq x^-$, and since $\{x, x^-, u, v\}$ does not form a claw in G, $vx^- \in E(G)$.

Suppose $\{a, b\} \neq \{v, x^-\}$ for some choice of $P \in \mathcal{P}_0$. Since $N_G(x)$ induces a 3-connected graph in G, there is a path Q which starts at $\{u, x^+\}$ and ends at $\{v, x^-\} - \{a, b\}$ in $G[N_G(x) - \{a, b\}]$. Choose such $P \in \mathcal{P}_0$ and the path Q so that Q is as short as possible. Since $\{uv, x^-x^+, x^-u, x^+v\} \subset N$, we can use $a\overrightarrow{P}x^-x^+\overrightarrow{P}uxv\overrightarrow{P}b$, $a\overrightarrow{P}xu\overleftarrow{P}x^+v\overrightarrow{P}b$ and $a\overrightarrow{P}x^-u\overleftarrow{P}xv\overrightarrow{P}b$ instead of P to switch the role of u and x^+. Similarly, we can switch the role of v and x^-. Therefore, we may assume Q starts at u and ends at v. Let $a_1 = u^{+(Q)}$. Note that $a_1 \in N_G(x)$ and hence $a_1v \in E(G')$. Furthermore, $a_1 \in V(P)$ by Lemma 5. First, suppose $a_1 \in x^+\overrightarrow{P}u$. By the minimality of Q, $a_1 \in x^{++}\overrightarrow{P}u^-$. This implies $u \neq x^+$. If $a_1^{+(P)}a_1^{-(P)} \in E(G)$, let $P' = a\overrightarrow{P}x^-xx^+\overrightarrow{P}a_1^{-(P)}a_1^{+(P)}\overrightarrow{P}ua_1v\overrightarrow{P}b$. If $a_1^{+(P)}x \in E(G)$, let $P' = a\overrightarrow{P}x^-xa_1^{+(P)}\overrightarrow{P}ux^+\overrightarrow{P}a_1v\overrightarrow{P}b$. If $a_1^{-(P)}x \in E(G)$, then let $P' = a\overrightarrow{P}x^-xa_1^{-(P)}\overleftarrow{P}x^+u\overleftarrow{P}a_1v\overrightarrow{P}b$. Then in each case $P' \in \mathcal{P}$, $|E(P') \cap N| \leq 1$ and $a_1\overrightarrow{Q}v$ is shorter than Q. This contradicts the choice of (P, Q). Thus, $\{a_1^{+(P)}, a_1^{-(P)}, x\} \subset N_G(a_1)$ is an independent set. Since G is claw-free, this is a contradiction. Therefore, we have $a_1 \notin x^+\overrightarrow{P}u$. This implies $a_1 \in a\overrightarrow{P}x^- \cup v\overrightarrow{P}b$.

Suppose $a_1 \in a\overrightarrow{P}x^-$. Since $a \notin V(Q)$ and $x^- \notin N_G(u)$, $a_1 \in a^+\overrightarrow{P}x^{--}$. If $a_1^{-(P)}a_1^{+(P)} \in E(G)$, let $P' = a\overrightarrow{P}a_1^{-(P)}a_1^{+(P)}x^-xx^+\overrightarrow{P}ua_1v\overrightarrow{P}b$. Then $P' \in \mathcal{P}$, $|E(P') \cap N| \leq 1$ and $a_1\overrightarrow{Q}v$ is shorter than Q. If $a_1^{+(P)}x \in E(G)$, then $a\overrightarrow{P}a_1u\overleftarrow{P}x^+xa_1^{+(P)}\overrightarrow{P}x^-v\overrightarrow{P}b$ is an ab-path of length m in G. If $a_1^{-(P)}x \in E(G)$, then $a\overrightarrow{P}a_1^{-(P)}xx^+\overrightarrow{P}ua_1\overrightarrow{P}x^-v\overrightarrow{P}b$ is an ab-path of length m in G. Hence we have a contradiction in each case. Therefore, $\{a_1^{-(P)}, a_1^{+(P)}, x\} \subset N_G(a_1)$ is an independent set in G. Since G is claw-free, this is a contradiction. If $a_1 \in v\overrightarrow{P}b$, we have a contradiction by similar arguments.

Therefore, we have $\{a, b\} = \{x^-, v\}$ for each $P \in \mathcal{P}_0$. Then $a = x^-$ and $b = v$. We consider a path Q in $N_G(x)$ from $\{x^+, u\}$ to $\{a, b\}$. Choose $P \in \mathcal{P}_0$ and Q so that Q is as short as possible. We may assume Q starts at u and ends at b. Let $a_1 = u^{+(Q)}$. Since $\{a, b\} \cap N_G(u) = \emptyset$, $a_1 \notin \{a, b\}$. Thus, $a_1 \in x^+\overrightarrow{P}u$. Then we have a contradiction by applying the same arguments as in the previous paragraph to $\{x, a_1^{-(P)}, a_1^{+(P)}\} \subset N_G(a_1)$. $\square$

A sequence $L = x_0x_1 \ldots x_l$ of vertices is said to be a *lollipop* if

(1) $x_0x_1 \ldots x_{l-1}$ is a path, and

(2) $x_{l-1}x_l \in E(G)$ and $x_l = x_i$ for some i, $1 \leq i \leq l-2$.

We say L starts at x_0 and ends at x_l. We also say that l is the length of L. We call the subsequence $x_ix_{i+1} \ldots x_l$ the *candy* of L, and the path $x_0, x_1, \ldots, x_i$ the *stick* of L, respectively. If $i < l-2$ the candy is a cycle. If $i = l-2$, then the candy is just one edge traversed twice.

In this case we say that L is a *trivial lollipop.*

Lemma 8. *Let G be a claw-free graph and let x and a be distinct vertices of G. Suppose $N_G(x)$ induces a 3-connected graph in G, and let G' be the graph obtained from G by local completion at x. If G' has an ax-path of length l, then G has an ax-path of length at least l.*

Proof. Let m be the length of a longest ax-path in G'. Then $m \geq l$. We prove that G has an ax-path of length at least m. Assume G has no ax-path of length at least m. Let $N = E(G') - E(G)$. By the assumption and Lemma 6, there exists a longest ax-path P in G' with $|E(P) \cap N| = 1$. Let $E(P) \cap N = \{uv\}$ with $u = v^-$. Then $\{u, v\} \subset N_G(x)$ and $a\overrightarrow{P}ux\overleftarrow{P}vx$ is a lollipop of length $m + 1$ in G which starts at a and ends at x. Let $\mathcal{L}$ be the set of all the lollipops of length $m + 1$ starting at a and ending at x in G. We consider two cases.

Case 1. There exists a nontrivial lollipop in $\mathcal{L}$.

Let $L \in \mathcal{L}$ be a nontrivial lollipop. Let P be the stick of L and C be the candy of L. Let $u = x^{-(P)}$, $v = x^{+(C)}$ and $w = x^{-(C)}$. Since L is nontrivial, $w \neq v$. If $uv \in E(G)$, then $a\overrightarrow{P}uv\overrightarrow{C}x$ is an ax-path of length m in G. This contradicts the assumption. Therefore, $uv \notin E(G)$. Similarly, $uw \notin E(G)$. Since $\{x, u, v, w\}$ does not form a claw in G, $vw \in E(G)$. We consider two subcases.

Subcase 1.1 $a \neq u$ for some choice of a nontrivial lollipop $L \in \mathcal{L}$.

Since $N_G(x)$ induces a 3-connected graph, there exists a path Q which joins v and u in $N_G(x) - \{a, w\}$. Furthermore, there exists a path R which joins w and u in $N_G(x) - \{a, v\}$. Choose such L, Q and R so that

(1) Q is as short as possible, and

(2) R is as short as possible, subject to (1).

Since uv, $uw \notin E(G)$, both Q and R have length at least two. On the other hand, since Q and R are induced paths in $N_G(x)$, both of them have length at most three by Lemma 4. Since $a\overrightarrow{P}uv\overrightarrow{C}wx$ is an ax-path of length m in G' and $uv \in E(G') - E(G)$, $N_G(x) \subset V(L)$ by Lemma 5. Therefore, $V(Q) \cup V(R) \subset V(L)$. However, possibly $V(Q) \cap V(R) - \{u\} \neq \emptyset$.

Let $a_1 = v^{+(Q)}$. Then $a_1 \in v\overrightarrow{C}w$ or $a_1 \in a\overrightarrow{P}u^-$.

Claim 1. $a_1 \in a\overrightarrow{P}u^-$

Proof. Assume $a_1 \in v\overrightarrow{C}w$. Since $a_1 \notin \{v, w\}$, $a_1 \in v^+\overrightarrow{C}w^-$. If $a_1^{+(C)}a_1^{-(C)} \in E(G)$, let

$C' = xa_1v\overrightarrow{C}a_1^{-(C)}a_1^{+(C)}\overrightarrow{C}x$. Then $P \cup C' \in \mathcal{L}$, $\{x^{+(C')}, x^{-(C')}\} = \{w, a_1\}$ and $a_1\overrightarrow{Q}u$ is shorter than Q and avoids w. This contradicts the choice of (L, Q). If $a_1^{+(C)}x \in E(G)$, then let $C' = xa_1\overleftarrow{C}vw\overleftarrow{C}a_1^{+(C)}x$. Then $P \cup C' \in \mathcal{L}$ and $\{x^{+(C')}, x^{-(C')}\} = \{a_1, a_1^{+(C)}\}$. Since $a_1\overrightarrow{Q}u$ is shorter than Q, this contradicts the choice of (L, Q) if $a_1^{+(C)} \notin V(Q)$. Thus, we have $a_1^{+(C)} \in V(Q)$. Since the length of Q is at most three, this implies $Q = va_1a_1^{+(C)}u$. Then $a\overrightarrow{P}ua_1^{+(C)}\overrightarrow{C}wv\overrightarrow{C}a_1x$ is an ax-path of length m in G. This is a contradiction. By a similar argument we have a contradiction if $a_1^{-(C)}x \in E(G)$. Thus, $\{x, a_1^{+(C)}, a_1^{-(C)}\}$ is an independent set in G. Since $\{x, a_1^{+(C)}, a_1^{-(C)}\} \subset N_G(a_1)$ and G is claw-free, this is a contradiction. Therefore, the claim is proved. $\square$

Claim 2. *$a_1^{-(P)}a_1^{+(P)} \notin E(G)$, $a_1^{+(P)}v \notin E(G)$ and $a_1^{-(P)}v \in E(G)$*

Proof. First, note $a_1^{-(P)}$ and $a_1^{+(P)}$ exist since $a_1 \notin \{a, x\}$. Suppose $a_1^{-(P)}a_1^{+(P)} \in E(G)$. Then let $P' = a\overrightarrow{P}a_1^{-(P)}a_1^{+(P)}\overrightarrow{P}x$ and $C' = xa_1v\overrightarrow{C}x$. Then $P' \cup C' \in \mathcal{L}$ and $\{x^{+(C')}, x^{-(C')}\} = \{a_1, w\}$. Since $a_1\overrightarrow{Q}u$ is shorter than Q and avoids w, this contradicts the choice of (L, Q).

Next, suppose $a_1^{+(P)}v \in E(G)$. If $a_1w \in E(G)$, then $a\overrightarrow{P}a_1w\overleftarrow{C}va_1^{+(P)}\overrightarrow{P}x$ is an ax-path of length m in G. If $a_1u \in E(G)$, then $a\overrightarrow{P}a_1u\overleftarrow{P}a_1^{+(P)}v\overrightarrow{C}wx$ is an ax-path of length m in G. Both contradict the assumption, and hence a_1w, $a_1u \notin E(G)$. Then since $uw \notin E(G)$ and $\{u, w, a_1\} \subset N_G(x)$ (note $u \neq a_1$), $\{x, u, w, a_1\}$ forms a claw in G. This is a contradiction. Therefore, $a_1^{+(P)}v \notin E(G)$.

Since $\{a_1^{-(P)}, a_1^{+(P)}, v\} \subset N_G(a_1)$, $a_1^{-(P)}a_1^{+(P)} \notin E(G)$, and $a_1^{+(P)}v \notin E(G)$, we have $a_1^{-(P)}v \in E(G)$. $\square$

Claim 3. *$a_1u \in E(G)$*

Proof. If $a_1w \in E(G)$, then $a\overrightarrow{P}a_1^{-(P)}v\overrightarrow{C}wa_1\overrightarrow{P}x$ is an ax-path of length m in G, a contradiction. Since $uw \notin E(G)$ and $\{x, a_1, w, u\}$ does not form a claw, we have $a_1u \in E(G)$. $\square$

Since $a_1u \in E(G)$, we have $Q = va_1u$.

Let $b_1 = w^{+(R)}$ (possibly $b_1 \in V(Q)$). Then $b_1 \in a^+\overrightarrow{P}u^-$ or $b_1 \in v^+\overrightarrow{C}w^-$. Because of the nonsymmetric choice of Q and R, the proof of the next claim is different from that of Claim 1.

Claim 4. *$b_1 \in a^+\overrightarrow{P}u^-$*

Proof. Assume $b_1 \in v^+\overrightarrow{C}w^-$. First, we claim $b_1^{+(C)}x \in E(G)$. Assume the contrary. Since $\{b_1, x, b_1^{+(C)}, b_1^{-(C)}\}$ does not form a claw in G, $b_1^{+(C)}b_1^{-(C)} \in E(G)$ or $xb_1^{-(C)} \in E(G)$. If

$b_1^{+(C)}b_1^{-(C)} \in E(G)$, then let $C' = xv\overrightarrow{C}b_1^{-(C)}b_1^{+(C)}\overrightarrow{C}wb_1x$. Then $P \cup C' \in \mathcal{L}$ and $\{x^{-(C')}, x^{+(C')}\}$ $= \{v, b_1\}$. Since $Q = va_1u$ avoids b_1 and $b_1\overrightarrow{R}u$ is shorter than R, this contradicts the choice of (L, Q, R). Thus, we have $xb_1^{-(C)} \in E(G)$. Consider $\{x, u, w, b_1^{-(C)}\}$. Since $uw \notin E(G)$, we have $ub_1^{-(C)} \in E(G)$ or $wb_1^{-(C)} \in E(G)$. If $ub_1^{-(C)} \in E(G)$, then $a\overrightarrow{P}ub^{-(C)}\overleftarrow{C}vw\overleftarrow{C}b_1x$ is an ax-path of length m in G, a contradiction. If $wb_1^{-(C)} \in E(G)$, let $C' = xv\overrightarrow{C}b_1^{-(C)}w\overleftarrow{C}b_1x$. Then $P \cup C' \in \mathcal{L}$ and $\{x^{-(C')}, x^{+(C')}\} = \{v, b_1\}$. Since $Q = va_1u$ avoids b_1 and $b_1\overrightarrow{R}u$ is shorter than R, this contradicts the choice of (L, Q, R). Therefore, we have $b_1^{+(C)}x \in E(G)$.

If $ub_1^{+(C)} \in E(G)$, then $a\overrightarrow{P}ub_1^{+(C)}\overrightarrow{C}wv\overrightarrow{C}b_1x$ is an ax-path of length m in G, a contradiction. Therefore, since $uv \notin E(G)$ and $\{x, u, v, b^{+(C)}\}$ does not form a claw in G, we have $vb_1^+ \in E(G)$.

If $a_1^{-(P)}v^{+(C)} \in E(G)$, then $a\overrightarrow{P}a_1^{-(P)}v^{+(C)}\overrightarrow{C}wva_1\overrightarrow{P}ux$ is an ax-path of length m in G, a contradiction. If $a_1^{-(P)}b_1^{+(C)} \in E(G)$, then $a\overrightarrow{P}a_1^{-(P)}b_1^{+(C)}\overrightarrow{C}wb_1\overleftarrow{C}va_1\overrightarrow{P}ux$ is an ax-path of length m, a contradiction. If $v^{+(C)}b_1^{+(C)} \in E(G)$, let $C' = xvw\overleftarrow{C}b_1^+v^{+(C)}\overrightarrow{C}b_1x$. Then $P \cup C' \in \mathcal{L}$ is a lollipop of length l and $\{x^{-(C')}, x^{+(C')}\} = \{v, b_1\}$. Since $Q = va_1u$ avoids b_1 and b_1Ru is shorter than R, this contradicts the choice of (L, Q, R). Therefore, $\{a_1^{-(P)}, v^{+(C)}, b_1^{+(C)}\} \subset N_G(v)$ is an independent set in G. Since G is claw-free, this is a contradiction. Therefore, the claim follows. $\square$

Claim 5. *$b_1^{-(P)}b_1^{+(P)} \notin E(G)$, $b_1^{+(P)}w \notin E(G)$, $b_1^{-(P)}w \in E(G)$ and $b_1u \in E(G)$.*

Proof. Note $b_1^{-(P)}$ and $b_1^{+(P)}$ exist since $b_1 \notin \{a, x\}$. We first prove $b_1^{-(P)}b_1^{+(P)} \notin E(G)$. This trivially follows from Claim 2 if $a_1 = b_1$. Hence we may assume $a_1 \neq b_1$. Assume $b_1^{-(P)}b_1^{+(P)} \in E(G)$. Let $P' = a\overrightarrow{P}b_1^{-(P)}b_1^{+(P)}\overrightarrow{P}x$ and $C' = xv\overrightarrow{C}wb_1x$. Then $P' \cup C' \in \mathcal{L}$ and $\{x^{+(C')}, x^{-(C')}\} = \{v, b_1\}$. Since $a_1 \neq b_1$, $Q = va_1u$ avoids b_1. Since $b_1\overrightarrow{R}u$ is shorter than R, this contradicts the choice of (L, Q, R). Thus, we have $b_1^{-(P)}b_1^{+(P)} \notin E(G)$.

Once we have $b_1^{-(P)}b_1^{+(P)} \notin E(G)$, we obtain $b_1^{+(P)}w \notin E(G)$, $b_1^{-(P)}w \in E(G)$ and $b_1u \in E(G)$ by the same arguments as those in the proofs of Claim 2 and Claim 3. $\square$

If $a_1 \in a\overrightarrow{P}b_1^{-(P)}$, then $a\overrightarrow{P}a_1^{-(P)}v\overrightarrow{C}wb_1^{-(P)}\overleftarrow{P}a_1u\overrightarrow{P}b_1x$ is an ax-path of length m. If $a_1 \in b_1^{+(P)}\overrightarrow{P}u^-$, then $a\overrightarrow{P}b_1^{-(P)}w\overleftarrow{C}va_1^{-(P)}\overleftarrow{P}b_1u\overleftarrow{P}a_1x$ is an ax-path of length m in G. Finally, if $a_1 = b_1$, then $a\overrightarrow{P}a_1^{-(P)}v\overrightarrow{C}wa_1\overrightarrow{P}x$ is an ax-path of length m in G. Therefore, we have a contradiction in each case, and the proof is complete in this subcase.

Subcase 1.2 $a = u$ for any choice of a nontrivial lollipop $L \in \mathcal{L}$.

There exists a path Q from $\{v,w\}$ to a in $G[N_G(x)]$. Choose nontrivial $L \in \mathcal{L}$ and Q so that Q is as short as possible. We may assume Q starts at v. Since $av\overrightarrow{C}x$ is an ax-path of length m in G', $V(Q) \subset V(L)$ by Lemma 5. Let $a_1 = v^{+(Q)}$. Then $a_1 \in v^+\overrightarrow{C}w^-$. If $a_1^{-(C)}a_1^{+(C)} \in E(G)$, let $C' = xa_1v\overrightarrow{C}a_1^{-(C)}a_1^{+(C)}\overrightarrow{C}wx$. If $a_1^{+(C)}x \in E(G)$, let $C' = xa_1\overleftarrow{C}vw\overleftarrow{C}a_1^{+(C)}x$. If $a_1^{-(C)}x \in E(G)$, let $C' = xa_1\overrightarrow{C}wv\overrightarrow{C}a_1^{-(C)}x$. Then in each case $ax \cup C' \in \mathcal{L}$ and $a_1\overrightarrow{Q}a$ is shorter than Q. This contradicts the choice of (L,Q).

Case 2. All the lollipops in $\mathcal{L}$ are trivial.

Since G has no ax-path of length at least m, $m \geq 2$. Let $L \in \mathcal{L}$ and let P be the stick of L and xv be the candy of L. Let $u = x^{-(P)}$. If $uv \in E(G)$, then $a\overrightarrow{P}uvx$ is an ax-path of length m in G, a contradiction. Thus, $uv \notin E(G)$. If $m = 2$, then $a = u$ and hence $a \in N_G(x)$. Then since $N_G(x)$ induces a connected graph in G, there exists a vu-path in $G[N_G(x)]$, and hence G has an ax-path of length at least two. Therefore, $m \geq 3$. In particular, $a \neq u$.

Since x is a locally 3-connected vertex, there exists a vu-path Q in $G[N_G(x)] - \{a, u^{-(P)}\}$ (or $G[N_G(x)] - a$ if $a = u^{-(P)}$). Let $a_1 = v^{+(Q)}$. Since $a\overrightarrow{P}uvx$ is an ax-path of length m in G', $V(Q) \subset V(L)$ by Lemma 5 and hence $a_1 \in V(P) - \{a, u, u^{-(P)}\}$. Consider $\{a_1, a_1^{-(P)}, a_1^{+(P)}, x\}$. If $a_1^{-(P)}a_1^{+(P)} \in E(G)$, then let $P' = a\overrightarrow{P}a_1^{-(P)}a_1^{+(P)}\overrightarrow{P}ux$ and $C' = xva_1x$. If $a_1^-x \in E(G)$, then let $P' = a\overrightarrow{P}a_1^{-(P)}x$ and $C' = xva_1\overrightarrow{P}ux$. Suppose $a_1^{+(P)}x \in E(G)$. Since $a_1 \neq u^{-(P)}$, $a_1^{+(P)} \neq u$. Let $P' = a\overrightarrow{P}a_1vx$ and $C' = xa_1^+\overrightarrow{P}ux$. Then in each case $P' \cup C' \in \mathcal{L}$ and it is nontrivial. This contradicts the assumption of the case, and the lemma is proved. $\square$

3. Homogeneous Traceability.

We prove Theorem 1 (2) by using the same proof strategy as that given in Section 2. A path starting at a vertex v is said to be a v-path. We prove the following theorem, which is similar to Theorem 3 in Section 2.

Theorem 9. *Let G be a claw-free graph and let x, $a \in V(G)$. Suppose $N_G(x)$ induces a 2-connected graph in G. Let G' be the graph obtained from G by local completion at x. If G' has an a-path of length l, then G has an a-path of length l.*

Actually, its proof is almost the same as those of Lemmas 7 and 8 in Section 2.

Proof. Let m be the length of a longest a-path in G'. Then $m \geq l$. We prove that G has an a-path of length at least m. Assume, to the contrary, G has no a-paths of length at least m. Let $N = E(G') - E(G)$. Let $\mathcal{P}$ be the set of all the longest a-paths in G'. If $P \in \mathcal{P}$, then $\{x\} \cup N_G(x) \subset V(P)$ by Lemma 5 since P is a longest ab-path for some $b \in V(G) - \{a\}$. Furthermore, by the assumption and Lemma 6, $|E(P) \cap N| = 1$ for some $P \in \mathcal{P}$. Let $\mathcal{P}_0 = \{P \in \mathcal{P} : |E(P) \cap N| = 1\}$.

Let $P \in \mathcal{P}_0$ and let $E(P) \cap N = \{uv\}$ with $u = v^-$. Let b be the terminal vertex of P. First, suppose $a \neq x$. If $x = b$, then $a\overrightarrow{P}ux\overleftarrow{P}v$ is an a-path of length m in G. This contradicts the assumption. Therefore, $x \in a^+\overrightarrow{P}u^- \cup v^+\overrightarrow{P}b^-$. Suppose $x \in a^+\overrightarrow{P}u^-$. Then by the same arguments as in the proof of Lemma 7, we have $\{x^-x^+, x^-u, vx^+\} \cap E(G) = \emptyset$, $vx^- \in E(G)$, and $ux^+ \in E(G)$ if $u \neq x^+$. Since $G[N_G(x)]$ is 2-connected, there exists a path Q with $V(Q) \subset N_G(x) - \{a\}$ which starts at $\{u, x^+\}$ and ends at $\{v, x^-\} - \{a\}$. Now choose $P \in \mathcal{P}_0$ and Q so that Q is as short as possible. Since we can use $a\overrightarrow{P}x^-x^+\overrightarrow{P}uxv\overrightarrow{P}b$ instead of P to switch the role of x^+ and u, we may assume Q starts at u. Let $a_1 = u^{+(Q)}$. If $a_1 \neq b$, then we can follow the same arguments as in the proof of Lemma 7, and obtain a contradiction. If $a_1 = b$, then $a\overrightarrow{P}ub\overleftarrow{P}v$ is an a-path of length m in G. This is a contradiction. We reach a contradiction by similar arguments if $x \in v^+\overrightarrow{P}b^-$.

Next, suppose $a = x$. Then $b\overleftarrow{P}vx\overrightarrow{P}ux$ is a lollipop of length $m+1$ which starts at b and ends at x in G. Let $\mathcal{L}$ be the set of all the lollipops of length $m+1$ starting at b and ending at x.

Suppose $\mathcal{L}$ has a nontrivial lollipop L. Let P and C be the stick and the candy of L, respectively, and $u = x^{-(P)}$, $v = x^{+(C)}$ and $w = x^{-(C)}$. By the same argument as in the proof of Lemma 8, we have $\{uv, uw\} \cap E(G) = \emptyset$ and $vw \in E(G)$.

If $b \neq u$ for some choice of $L \in \mathcal{L}$, choose such L, a path Q in $N_G(x) - \{w\}$ which joins v and u, and a path R in $N_G(x) - \{v\}$ which joins w and u so that

(1) Q is as short as possible, and

(2) R is as short as possible, subject to (1).

Let $a_1 = v^{+(Q)}$ and $b_1 = w^{+(R)}$. If $b \notin \{a_1, b_1\}$, we can follow the same arguments as in Subcase 1.1 of the proof of Lemma 8, and obtain a contradiction. If $a_1 = b$, then $xw\overleftarrow{C}vb\overrightarrow{P}u$ is a path of length m in G. If $b_1 = b$, then $xv\overrightarrow{C}wb\overrightarrow{P}u$ is a path of length m in G. Therefore, we have a contradiction in each case. If $b = u$ for any choice of $L \in \mathcal{L}$, then we can follow the same arguments as in Subcase 1.2 of the proof of Lemma 8 to obtain a contradiction.

Suppose all the lollipops in $\mathcal{L}$ are trivial. Then $m \geq 2$. Let P be the stick of L and let xv

be the candy of L. Let $u = x^{-(P)}$. Then by the same argument as in the proof of Lemma 8, we have $uv \notin E(G)$, $m \geq 3$ and $b \neq u$. Let Q be a vu-path in $G[N_G(x) - \{u^{-(P)}\}]$ and let $a_1 = v^{+(Q)}$. If $a_1 \neq b$, we can follow the same arguments as in Case 2 of the proof of Lemma 8, and obtain a contradiction. If $a_1 = b$, then $x\overleftarrow{P}bv$ is a path of length m in G. This is a final contradiction, and the theorem follows. $\Box$

4. The Uniqueness of the Closure

In this section we consider the uniqueness of a closure in a more generalized situation. For a vertex x in a graph G we shall write G_x for the graph induced by $N_G(x)$. Let $G\langle x\rangle$ be the graph obtained from G by local completion at x. For $x_1, x_2, \ldots, x_r \in V(G)$, we write $G\langle x_1, x_2, \ldots, x_r\rangle$ for $G\langle x_1\rangle\langle x_2\rangle \ldots \langle x_r\rangle$. Given a property $\mathcal{P}$ of graphs, we shall say that G *can be completed at x with respect to $\mathcal{P}$* if G_x satisfies $\mathcal{P}$. Thus, if $\mathcal{P}_0$ is the property of being connected and noncomplete, an eligible vertex is a vertex which can be completed with respect to $\mathcal{P}_0$. A graph G is said to be $\mathcal{P}$-closed if G_x is complete for every vertex x with G_x satisfying $\mathcal{P}$. For $x_1, \ldots, x_r \in V(G)$, if $\overline{G} = G\langle x_1, \ldots, x_r\rangle$ is $\mathcal{P}$-closed, we shall say that $\overline{G}$ is a $\mathcal{P}$-closure of G. In this context, a k-closure is a $\mathcal{P}$-closure, where $\mathcal{P}$ is the property of k-connectedness.

If $\mathcal{P}$-closures are to be unique, then we expect that if G can be completed at either of x and y with respect to $\mathcal{P}$, then it can be completed at both – i.e. that $G\langle x\rangle$ can be completed at y with respect to $\mathcal{P}$. Motivated by this observation, we introduce the following definition.

For vertices $x_1, x_2, \ldots, x_r$ in G we shall say that $G\langle x_1, x_2, \ldots, x_r\rangle$ is *defined with respect to $\mathcal{P}$* if

$$\begin{cases} G \text{ can be completed at } x_1 \text{ with respect to } \mathcal{P} & \text{if } r = 1 \\ G\langle x_1, \ldots, x_{r-1}\rangle \text{ is defined with respect to } \mathcal{P} & \\ \qquad \text{and } G\langle x_1, \ldots, x_{r-1}\rangle \text{ can be completed at } x_r \text{ with respect to } \mathcal{P} & \text{if } r \geq 2. \end{cases}$$

If the property $\mathcal{P}$ is clear in the context, we sometimes omit "with respect to $\mathcal{P}$", and simply say that "$G\langle x_1, x_2, \ldots, x_r\rangle$ is defined". A property $\mathcal{P}$ of graphs G is said to be *well-behaved* if $G\langle x, y\rangle$ is defined whenever both $G\langle x\rangle$ and $G\langle y\rangle$ are defined with respect to $\mathcal{P}$.

If $\mathcal{P}$ is a well-behaved property of graphs and both $G\langle x\rangle$ and $G\langle y\rangle$ are defined with respect to $\mathcal{P}$, then both $G\langle x, y\rangle$ and $G\langle y, x\rangle$ are defined with respect to $\mathcal{P}$. Furthermore, due to the nature of local completion, they are the same.

For a set S, let K_S be the complete graph whose vertex set is S.

Lemma 10. *Let $\mathcal{P}$ be a property of graphs. For a graph G and x, $y \in V(G)$, if both $G\langle x, y\rangle$ and $G\langle y, x\rangle$ are defined, then $G\langle x, y\rangle = G\langle y, x\rangle$.*

Proof. If $xy \notin E(G)$, then

$$G\langle x, y\rangle = G \cup K_{N_G(x)} \cup K_{N_G(y)} = G \cup K_{N_G(y)} \cup K_{N_G(x)} = G\langle y, x\rangle.$$

If $xy \in E(G)$, then

$$G\langle x, y\rangle = G \cup K_{N_G(x)\cup N_G(y)\cup\{x,y\}} = G \cup K_{N_G(y)\cup N_G(x)\cup\{x,y\}} = G\langle y, x\rangle. \quad \square$$

Note that we do not assume that $\mathcal{P}$ is well-behaved in the above lemma. This lemma holds even if $\mathcal{P}$ is not well-behaved.

Now we prove the following theorem.

Theorem 11. *Let $\mathcal{P}$ be a well-behaved property of graphs, and let G be a graph and $x_1, \ldots, x_r, z_1, \ldots, z_s \in V(G)$. If both $G\langle x_1, \ldots, x_r\rangle$ and $G\langle z_1, \ldots, z_s\rangle$ are defined with respect to $\mathcal{P}$, then both $G\langle x_1, \ldots, x_r, z_1, \ldots, z_s\rangle$ and $G\langle z_1, \ldots, z_s, x_1, \ldots, x_r\rangle$ are defined, and they are the same.*

Proof. We proceed by induction on $r+s$. If $r = s = 1$, then the theorem follows by the assumption and Lemma 10. Suppose $r+s > 2$. By symmetry we may assume $r > 1$. By the assumption $G\langle x_1, \ldots, x_{r-1}\rangle$ is defined. Then by the induction hypothesis both $G\langle x_1, \ldots, x_{r-1}, z_1, \ldots, z_s\rangle$ and $G\langle z_1, \ldots, z_s, x_1, \ldots, x_{r-1}\rangle$ are defined and they are the same. Since $G\langle x_1, \ldots, x_{r-1}\rangle\langle x_r\rangle$ and $G\langle x_1, \ldots, x_{r-1}\rangle\langle z_1, \ldots, z_s\rangle$ are defined, we can apply the induction hypothesis again to see that $G\langle x_1, \ldots, x_{r-1}, x_r, z_1, \ldots, z_s\rangle$ and $G\langle x_1, \ldots, x_{r-1}, z_1, \ldots, z_s, x_r\rangle$ are defined, and they are the same. However, since

$$\begin{aligned} G\langle x_1, \ldots, x_{r-1}, z_1, \ldots, z_s, x_r\rangle &= G\langle x_1, \ldots, x_{r-1}, z_1, \ldots, z_s\rangle\langle x_r\rangle \\ &= G\langle z_1, \ldots, z_s, x_1, \ldots, x_{r-1}\rangle\langle x_r\rangle, \end{aligned}$$

$G\langle z_1, \ldots, z_s, x_1, \ldots, x_r\rangle$ is also defined, and

$$G\langle z_1, \ldots, z_s, x_1, \ldots, x_r\rangle = G\langle x_1, \ldots, x_{r-1}, z_1, \ldots, z_s, x_r\rangle = G\langle x_1, \ldots, x_r, z_1, \ldots, z_s\rangle \quad \square$$

The uniqueness of a well-behaved closure is deduced immediately from Theorem 11.

Corollary 12. *Let $\mathcal{P}$ be a well-behaved property of graphs. Let G be a graph and $x_1, \ldots, x_r$, $z_1 \ldots, z_s \in V(G)$. If both $G_1 = G\langle x_1, \ldots, x_r\rangle$ and $G_2 = G\langle z_1, \ldots, z_s\rangle$ are $\mathcal{P}$-closures of G, then $G_1 = G_2$.*

Proof. First, note that if $H_1 \subset H_2$ and both $H_1\langle y_1, \ldots, y_t\rangle$ and $H_2\langle y_1, \ldots, y_t\rangle$ are defined for $y_1, \ldots, y_t \in V(H_1)$, then $H_1\langle y_1, \ldots, y_t\rangle \subset H_2\langle y_1, \ldots, y_t\rangle$.

By Theorem 11, $G_1\langle z_1, \ldots, z_s\rangle$ is defined, and since $G \subset G_1$, $G_2 = G\langle z_1, \ldots, z_s\rangle \subset G_1\langle z_1, \ldots, z_s\rangle$. However, since G_1 is $\mathcal{P}$-closed, $G_1\langle z_1, \ldots, z_s\rangle = G_1$. Hence we have $G_2 \subset G_1$. By symmetry, we also have $G_1 \subset G_2$, and hence $G_1 = G_2$. $\square$

Let H and H' be graphs. If either (a) H' is obtained from H by adding edges (i.e. H is a spanning subgraph of H'), or (b) $H' = H \cup K_S$ for some $S \subset V(H')$ (possibly $S \not\subset V(H)$) with $N_H(x) \subset S$ for some $x \in V(H) \cap S$, then we shall say that H' is an *extension* of H. We shall also say that a property $\mathcal{P}$ is extendable if $\mathcal{P}$ is closed under extension. More precisely, suppose $\mathcal{P}$ satisfies the following condition.

($*$) *If H is a graph satisfying $\mathcal{P}$ and H' is an extension of H, then H' satisfies $\mathcal{P}$.*

Then $\mathcal{P}$ is said to be an *extendable* property.

Theorem 13.

(1) *Every extendable property is well-behaved.*

(2) *Let $\mathcal{P}$ be an extendable property of graphs and let $\mathcal{Q}$ be a property of graphs. If H is a $\mathcal{Q}$-closure of a graph G and G can be completed at x with respect to $\mathcal{P}$, then H can be completed at x with respect to $\mathcal{P}$.*

Proof. First, we prove (1). Let x, $y \in V(G)$ and let $\mathcal{P}$ be an extendable property of graphs. Suppose G can be completed at y with respect to $\mathcal{P}$. Then since $G\langle x\rangle_y$ is an extension of G_y and $\mathcal{P}$ is extendable, $G\langle x\rangle$ can be completed at y with respect to $\mathcal{P}$. Thus, $\mathcal{P}$ is well-behaved.

In order to prove (2), we first note that in the above argument we do not assume G can be completed at x. Thus, if G can be completed at x with respect to $\mathcal{P}$ and $H = G\langle y_1, \ldots, y_t\rangle$ is a $\mathcal{Q}$-closure, then H_x is obtained from G_x by a series of extensions and hence H can be completed at x with respect to $\mathcal{P}$. $\square$

Now we give a proof of Theorem 1 (1) as a corollary of Theorem 13.

Corollary 14. *The k-closure of a graph is uniquely determined for each k.*

Proof. Let $\mathcal{C}_k$ be the property of being k-connected, and we prove that $\mathcal{C}_k$ is an extendable property. Then the uniqueness of the k-closure follows from Theorem 13 (1) and Corollary 12. Let G be a k-connected graph and let G' be an extension of G. If G' is obtained from G by adding edges, then clearly G' is k-connected. Suppose G' is not obtained from G by edge-addition. Then $G' = G \cup K_S$ for some $S \subset V(G')$ with $N_G(x) \subset S$ for some $x \in V(G) \cap S$. Since G is k-connected, $|S \cap V(G)| \geq |N_G(x) \cup \{x\}| \geq k+1$. Thus, G' is also k-connected. $\square$

As another application of Theorem 13, we consider the property of having bounded independence number. Let $\mathrm{Ind}_{<r}$ be the property of having independence number less than r.

Theorem 15. *The property $\mathrm{Ind}_{<r}$ is extendable.*

Proof. Let G be a graph with $\alpha(G) < r$ and let G' be an extension of G. If G' is obtained from G by adding edges, then clearly $\alpha(G') < r$. Suppose $G' = G \cup K_S$ for some $S \subset V(G')$ with $N_G(x) \subset S$ for some $x \in V(G) \cap S$. Let T' be a maximum independent set of G'. Since $G'[S]$ is complete, $|T' \cap S| \leq 1$. On the other hand, since $N_G(x) \subset S$, $(T' - S) \cup \{x\}$ is an independent set of G. Therefore, $|T' - S| \leq \alpha(G) - 1$. Thus, $|T'| \leq \alpha(G)$, which implies $\alpha(G') \leq \alpha(G) < r$. $\square$

Let G be a $K_{1,r}$-free graph. Then G_x satisfies $\mathrm{Ind}_{<r}$ for each $x \in V(G)$. Since $\mathrm{Ind}_{<r}$ is an extendable property, we have the following corollary from Theorem 13 (2).

Corollary 16. *The k-closure of a $K_{1,r}$-free graph is $K_{1,r}$-free for each $k \geq 1$.* $\square$

Theorem A (1) corresponds to the case $r = 3$ in the above corollary.

In order to demonstrate the usefulness of the notion of extendable properties, we prove the following theorem on $\mathrm{Ind}_{<3}$-closure. Note that the $\mathrm{Ind}_{<3}$-closure is uniquely determined since the property $\mathrm{Ind}_{<3}$ is extendable.

Theorem 17. *Let G be a graph. Then G has a 1-factor if and only if the $\mathrm{Ind}_{<3}$-closure of G has a 1-factor.*

Proof. Since G is a spanning subgraph of its $\mathrm{Ind}_{<3}$-closure, the "only if" part is trivial. In order to prove the "if" part, we have only to prove the following statement.

(**) *Let G be a graph and let x be a vertex of G with $\alpha(G_x) < 3$. Let G' be the graph obtained from G by local completion at x. If G' has a 1-factor, then G has a 1-factor.*

Assume G has no 1-factor. Choose a 1-factor F of G' so that $|F \cap (E(G') - E(G))|$ is as small as possible. By the assumption $F \cap (E(G') - E(G)) \neq \emptyset$, say $ab \in F \cap (E(G') - E(G))$. Then $\{a, b\} \subset N_G(x)$. Since F is a 1-factor of G', $xy \in F$ for some $y \in V(G)$. Since $x \notin N_G(x)$, $xy \in E(G)$. Since $\alpha(G_x) < 3$, $\{a, b, y\} \subset N_G(x)$ and $ab \notin E(G)$, we have either $ay \in E(G)$ or $by \in E(G)$. By symmetry we may assume $by \in E(G)$. Let $F_0 = F - \{ab, xy\} \cup \{ax, by\}$. Then F_0 is a 1-factor in G' with $|F_0 \cap (E(G') - E(G))| = |F \cap (E(G') - E(G))| - 1$. This contradicts the minimality of $|F \cap (E(G') - E(G))|$. Thus, G has a 1-factor. $\Box$

If G is a connected claw-free graph, then G can be completed at every vertex with respect to $\mathrm{Ind}_{<3}$. Since $\mathrm{Ind}_{<3}$ is an extendable property, we can apply Theorem 13 (2) with $\mathcal{P} = \mathcal{Q} = \mathrm{Ind}_{<3}$ to see that the $\mathrm{Ind}_{<3}$-closure of G is complete. Therefore, we have the following result by Sumner [4] as an immediate corollary.

Corollary 18 ([4])**.** *Every connected claw-free graph of even order has a 1-factor.* $\Box$

5. Concluding Remarks.

In Theorem 1 (3), we cannot replace "$cl_2(G)$" by "$cl_1(G)$" because of Theorem C (3). On the other hand, in Theorem 1 (2), we have no claw-free graph G such that $cl_2(G)$ is hamiltonian-connected while G is not hamiltonian-connected. Actually, we believe in the following conjecture.

Conjecture 19. *Let G be a claw-free graph. Then G is hamiltonian-connected if and only if $cl_2(G)$ is hamiltonian-connected.*

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