Forbidden subgraphs, stability and hamiltonicity

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Abstract

We study the stability of some classes of claw-free graphs defined in terms of forbidden subgraphs under the closure operation defined in [10]. We characterize all connected graphs A such that the class of all CA-free graphs (where C denotes the claw) is stable. Using this result, we prove that every 2-connected and CHP_8-free, CHZ_5-free or $CHN_{1,1,4}$-free graph is either hamiltonian or belongs to some classes of exceptional graphs (all of them having connectivity 2).

1 Introduction

In this paper we follow up with the considerations that originated in the papers [2] and [5]. All graphs considered here will be finite undirected graphs $G = (V(G), E(G))$ without loops and multiple edges. For terminology and notation not defined here we refer to [1]. We recall here briefly some basic concepts and notations from [5].

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If $A \subset V(G)$, then the induced subgraph on A in G will be denoted by $\langle A \rangle_G$ (or simply by $\langle A \rangle$). By a *clique* we mean a (not necessarily maximal) complete subgraph of G. We denote by P_k ($k \geq 2$) the path on k vertices, i.e. of length $k-1$. For $A, B \subset V(G)$, a path in G having one endvertex in A and the other in B will be referred to as an A,B-path. The *circumference* of G (i.e. the length of a longest cycle in G) is denoted by $c(G)$ and the *clique number* of G (i.e. the size of a largest clique in G) is denoted by $\omega(G)$.

If $H_1, \ldots, H_k (k \geq 1)$ are graphs, then a graph G is said to be $H_1, \ldots, H_k$*-free* if G contains no copy of any of the graphs $H_1, \ldots, H_k$ as an induced subgraph; the graphs $H_1, \ldots, H_k$ will be also referred to in this context as *forbidden subgraphs*. Specifically, the four-vertex star $K_{1,3}$ is also denoted by C and called the *claw* and in this case G is said to be *claw-free*. Whenever vertices of an induced claw are listed, its *center*, (i.e. its only vertex of degree 3) is always the first vertex of the list. Further graphs that will be often considered as forbidden subgraphs are shown in Fig. 1.

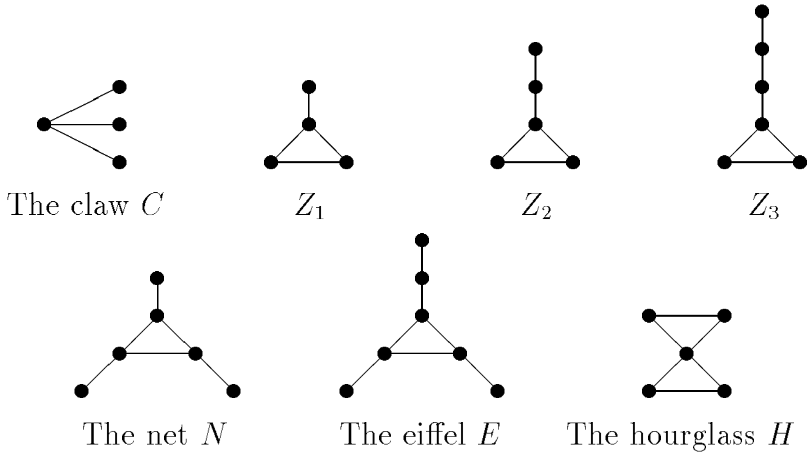


Figure 1

A vertex $x \in V(G)$ is said to be *locally connected* (*eligible, simplicial, locally disconnected*) if the subgraph $\langle N_G(x) \rangle$ is connected (connected noncomplete, complete, disconnected). The set of all locally connected (eligible, simplicial, locally disconnected) vertices of G is denoted by $V_{LC}(G)$ ($V_{EL}(G), V_{SI}(G), V_{LD}(G)$), respectively. Thus, the sets $V_{EL}(G), V_{SI}(G), V_{LD}(G)$ are pairwise disjoint, $V_{EL}(G) \cup V_{SI}(G) = V_{LC}(G)$ and $V_{LC}(G) \cup V_{LD}(G) = V(G)$.

For any $x \in V_{EL}(G)$ denote $B_x = \{uv|\ u, v \in N_G(x), uv \notin E(G)\}$ and let G'_x be the graph with vertex set $V(G'_x) = V(G)$ and with edge set $E(G'_x) = E(G) \cup B_x$ (i.e., G'_x is obtained from G by adding to $\langle N_G(x) \rangle_G$ all missing edges). The graph G'_x is called the *local completion of* G *at* x. It was proved in [10] that for any claw-free graph G and for any eligible vertex $x \in V_{EL}(G)$, the graph G'_x is claw-free and $c(G'_x) = c(G)$. The following concept was introduced in [10].

Let G be a claw-free graph. We say that a graph H is a closure of G, *denoted $H = \mathrm{cl}(G)$, if*

(i) there is a sequence of graphs $G_1, \ldots, G_t$ and vertices $x_1, \ldots, x_{t-1}$ such that $G_1 = G$, $G_t = H$, $x_i \in V_{EL}(G_i)$ and $G_{i+1} = (G_i)'_{x_i}$, $i = 1, \ldots, t-1$,

(ii) $V_{EL}(H) = \emptyset$.

(Equivalently, $\mathrm{cl}(G)$ is obtained from G by recursively repeating the operation of local completion, as long as this is possible). The following result summarizes basic properties of the closure operation.

Theorem A [10]. *Let G be a claw-free graph. Then*

(i) the closure $\mathrm{cl}(G)$ is well-defined,

(ii) there is a triangle-free graph H such that $\mathrm{cl}(G)$ is the line graph of H,

(iii) $c(G) = c(\mathrm{cl}(G))$.

(Specifically, part *(i)* of Theorem A implies that $\mathrm{cl}(G)$ does not depend on the order of eligible vertices used during the construction of $\mathrm{cl}(G)$).

We say that a claw-free graph G is *closed* if $G = \mathrm{cl}(G)$. Thus, G is closed if and only if $V_{EL}(G) = \emptyset$.

Let $\mathcal{C}$ be a subclass of the class of claw-free graphs. Following [2], we say that the class $\mathcal{C}$ is *stable under the closure* (or simply *stable*) if $\mathrm{cl}(G) \in \mathcal{C}$ for every $G \in \mathcal{C}$. It is easy to see that the class of k-connected claw-free graphs is an example of a stable class for any $k \geq 1$. By Theorem A, both the class of hamiltonian claw-free graphs and the class of 2-connected nonhamiltonian claw-free graphs are stable.

In this paper we continue with the study of stability of classes of graphs defined in terms of forbidden subgraphs originated in [5]. In Section 2 we characterize all connected graphs A such that the class of CA-free graphs is stable. Using this result, in Section 3 we extend several previous results on hamiltonicity in classes of graphs defined in terms of forbidden subgraphs.

2 Forbidden pairs and stability

In the main result of this section, Theorem 4, we characterize all connected graphs A such that the class of CA-free graphs is stable. We first recall several known results and prove some auxiliary statements. We denote by:

Z_i $(i \geq 1)$ – the graph which is obtained by identifying a vertex of a triangle with an endvertex of a path of length i,

$N_{i,j,k}$ $(i,j,k \geq 1)$ – the *generalized* (i,j,k)*-net*, i.e. the graph which is obtained by identifying each vertex of a triangle with an endvertex of one of three vertex-disjoint paths of lengths i,j,k.

(See also Fig. 2.) Thus, $N_{1,1,1} \simeq N$ and $N_{1,1,2} \simeq E$. We will always keep the labelling of the vertices of the graphs Z_i and $N_{i,j,k}$ as shown in Figure 2.

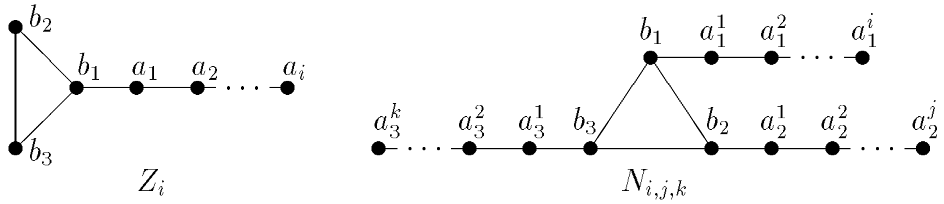


Figure 2

The following results were proved in [5].

Theorem B [5].
(i) Let G be a CP_i-free graph $(i \geq 3)$ and let $x \in V_{EL}(G)$. Then the graph G'_x is CP_i-free.
(ii) Let G be a $CN_{i,j,k}$-free graph $(i,j,k \geq 1)$ and let $x \in V_{EL}(G)$. Then the graph G'_x is $CN_{i,j,k}$-free.

Corollary C [5].
(i) The class of CP_i-free graphs is a stable class for any $i \geq 3$.
(ii) The class of $CN_{i,j,k}$-free graphs is a stable class for any $i,j,k \geq 1$.

It was also shown in [5] that the analogue of Theorem B fails in the class of CZ_i-free graphs, but the analogue of Corollary C still remains true in this class.

Theorem D [5]. *The class of CZ_i-free graphs is a stable class for any $i \geq 1$.*

The graph in Figure 3 is an example of a CH-free graph such that G'_x is not CH-free. Thus, the analogue of Theorem B fails also in the class of CH-free graphs. The following theorem shows that we can still prove the analogue of Corollary C in this case.

Theorem 1. *The class of CH-free graphs is a stable class.*

Proof. If F is an induced subgraph of G, then we say that F is a *permanent* (or *temporary*) *induced subgraph* of G if $\langle V(F)\rangle_{\mathrm{cl}(G)} \simeq F$ (or $\langle V(F)\rangle_{\mathrm{cl}(G)} \not\simeq F$), respectively.

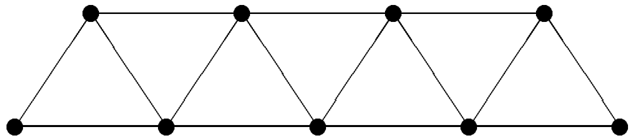


Figure 3

Let G be a CH-free graph and let $G_1, \ldots, G_t$ be the sequence of graphs that yields $\mathrm{cl}(G)$ (i.e. $G = G_1$, $\mathrm{cl}(G) = G_t$, $x_j \in V_{EL}(G_j)$ and $G_{j+1} = (G_j)'_{x_j}$, $j = 1, \ldots, t-1$). Suppose that $\mathrm{cl}(G)$ is not H-free and let $j_0 \geq 1$ be the smallest integer such that G_{j_0+1} contains a permanent induced H. Without loss of generality we can set $G_{j_0} = G$, i.e., we suppose that $\langle V(F)\rangle_{G'_x} = \langle V(F)\rangle_{\mathrm{cl}(G)} \simeq H$ for some $x \in V_{EL}(G)$ and $F \subset G'_x$, and that every induced H in G is temporary. Since $\langle V(F)\rangle_G \not\simeq H$ (otherwise F is a permanent H in G), we have $B_x^F = E(F) \cap B_x \neq \emptyset$.

Denote by z the center (i.e., the only vertex of degree four) of F and by a_1a_2, b_1b_2 the two edges in $N_G(z) \cap E(F)$. If both $xa_i \in E(G)$ and $xb_j \in E(G)$ for some $i, j \in \{1, 2\}$, then we have $a_ib_j \in E(G'_x)$, contradicting the assumption that $\langle V(F)\rangle_{G'_x} \simeq H$. Hence we can suppose without loss of generality that $B_x^F \subset \{b_1b_2,\ b_1z,\ b_2z\}$.

If both $zb_1 \in E(G)$ and $zb_2 \in E(G)$, then $b_1b_2 \in B_x$, implying $\langle\{z, b_1, b_2, a_1\}\rangle_G \simeq C$ – a contradiction. Hence we can assume that $zb_2 \in B_x$. Since $\langle V(F)\rangle_{G'_x} \simeq H$, we have $x \notin \{z, a_1, a_2, b_2\}$. Choose a shortest z, b_2-path in $\langle N_G(x)\rangle_G$ (it always exists since $x \in V_{EL}(G)$) and let c be its first vertex distinct from z (it exists since $zb_2 \notin E(G)$). Consider $F_1 = \langle\{z, a_1, a_2, x, c\}\rangle_G$. Since F_1 cannot be a permanent hourglass, we have $\{a_1x, a_2x, a_1c, a_2c\} \cap E(\mathrm{cl}(G)) \neq \emptyset$. But then, since $zb_2 \in E(G'_x)$, z becomes eligible in some G_j ($1 \leq j < t-1$), implying that $\langle V(F)\rangle_{\mathrm{cl}(G)}$ is a clique. This contradiction proves Theorem 1. ■

It is easy to observe that if G is CT-free (here T denotes the triangle), then G consists only of paths and cycles. Hence $V_{EL}(G) = \emptyset$, implying that $\mathrm{cl}(G) = G$ is CT-free. The class of CT-free graphs is thus also trivially stable. Furthermore, the class of CA-free graphs is also trivially stable if A contains an induced claw (since then C-free implies A-free) or if A is not closed (since then every closed graph is A-free). Hence we can without loss of generality restrict our observations to connected closed claw-free graphs A.

In the main result of this section, Theorem 4, we show that there are no other connected closed claw-free graphs A such that the CA-free class is stable except T and the graphs mentioned in Corollary C, Theorem D and Theorem 1. To prove this, we will need first some lemmas.

If G is a closed claw-free graph and $K \subset G$ is a maximal clique, then the vertices in $V(K) \cap V_{LD}(G)$ will be called the *vertices of attachment* of K.

Lemma 2. *Let A be a closed connected claw-free graph such that the class of CA-free graphs is stable. Then $\omega(A) \leq 4$.*

Proof. Let $q = \omega(A)$ and suppose, to the contrary, that $q \geq 5$. We define a graph G by the following construction.

Let $K^1, \ldots, K^s$ $(s \geq 1)$ be the collection of all maximum cliques of A (i.e., $K^i \simeq K_q$, $i = 1, \ldots, s$). Since G is closed, these cliques are pairwise edge-disjoint. For every i, $1 \leq i \leq s$, denote by $v_1, \ldots, v_{p_i}$ the vertices of attachment of K^i, choose an integer r_i such that $r_i \geq p_i$ and $p_i + 3r_i \geq q$, and replace the K^i in A by a copy of the graph L_{p_i,r_i} of Figure 4 in such a way that the vertices $v_1, \ldots, v_{p_i}$ of the copy of the L_{p_i,r_i} are identified with the vertices of attachment of the (removed) clique K^i. Then clearly G is claw-free, $\omega(G) \leq q-1$ and hence G is A-free; since every inserted copy of L_{p_i,r_i} closes up to a clique on $p_i + 3r_i \geq q$ vertices, it is straightforward to check that $\mathrm{cl}(G)$ contains an induced subgraph isomorphic to A. ∎

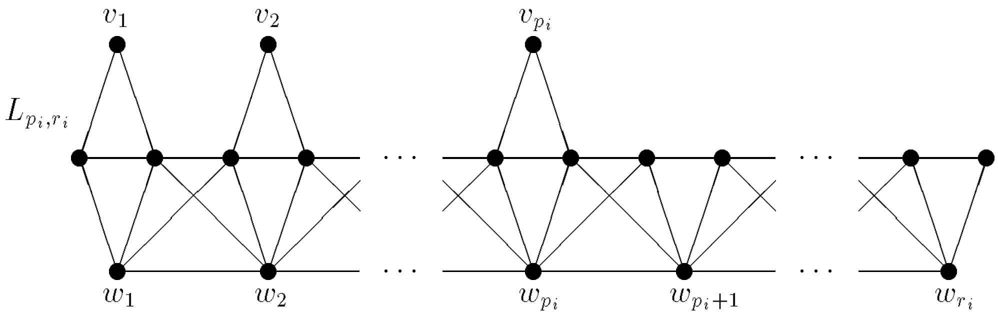


Figure 4

Lemma 3. *Let A be a closed connected claw-free graph such that the class of CA-free graphs is stable. Then $\omega(A) \leq 3$.*

Proof. By Lemma 2, $\omega(A) \leq 4$. Suppose, to the contrary, that A contains an induced K_4. We distinguish two cases.

Case 1: A contains an induced K_4 with $p \leq 3$ vertices of attachment,

Construct G by taking a copy of A and replacing each K_4 with one vertex of attachment by the graph L_1 of Figure 5 (where the vertex v is identified with the vertex of attachment of the K_4), each K_4 with two vertices of attachment by a copy of the graph L_2 of Fig. 5 and each K_4 with three vertices of attachment by a copy of the graph L_3 of Fig. 5 (where the replacement is again done in such a way that the vertices v_1, v_2 of the L_2 or v_1, v_2, v_3 of the L_3 are identified with the vertices of attachment of the deleted K_4). Then G is obviously claw-free and no closed induced subgraph of G contains a K_4 with at most three vertices of attachment. Hence G is CA-free. However, it is straightforward to check that $\mathrm{cl}(G)$ contains an induced A. ■

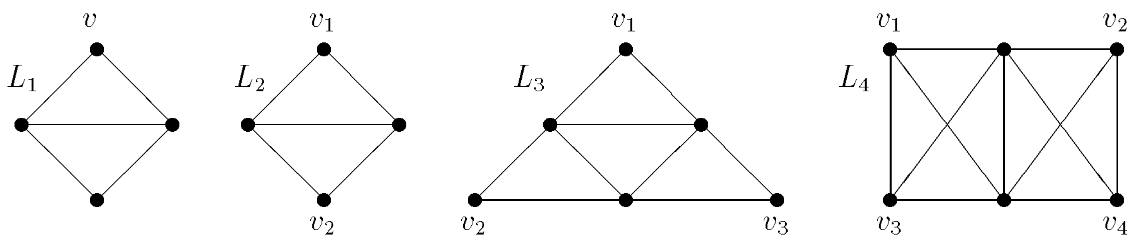


Figure 5

<u>Case 2:</u> Every induced K_4 in A has four vertices of attachment.

In this case we replace each K_4 in a copy of A by a copy of the graph L_4 of Fig. 5 (identifying again v_1, v_2, v_3, v_4 with the vertices of attachment of the K_4). The constructed graph G is again claw-free and, although the number of K_4's is not reduced this time, it is easy to see that no induced subgraph of G containing a K_4 with four vertices of attachment is closed. Hence G is CA-free. Since $\mathrm{cl}(G)$ contains an induced A, we have again a contradiction. ■

Now we are ready to prove the main result of this section.

Theorem 4. *Let A be a closed connected claw-free graph. Then the class of CA-free graphs is stable if and only if*

$$A \in \{H, T\} \cup \{P_i |\ i \geq 3\} \cup \{Z_i |\ i \geq 1\} \cup \{N_{i,j,k} |\ i, j, k \geq 1\}.$$

Proof. (i) The "if" part of the proof follows immediately from Corollary C, Theorem D, Theorem 1 and from the remarks before Lemma 2.

(ii) Suppose that the class of CA-free graphs is stable. By Lemma 3, $\omega(A) \leq 3$. If A is T-free, then A is either a path and we are done, or A is a cycle and then it is apparent that the CA-free class is not stable (since the graph G obtained from A by replacing one edge by the graph L_2 of Fig. 5 is CA-free while $\mathrm{cl}(G)$ is not). Hence we can suppose that A contains a triangle.

If A contains a triangle with two vertices of attachment, then, replacing in a copy of A every such triangle by a copy of the graph L_2 of Fig. 5 we obtain a CA-free graph G such that $\mathrm{cl}(G)$ is not CA-free; thus A has no triangle with two vertices of attachment.

If A contains a triangle with no vertex of attachment, then, since A is connected, $A \simeq T$ and we are done. Hence we can suppose that every triangle in A has one or three vertices of attachment.

We now show that if A contains a triangle with three vertices of attachment, then A contains no other triangle. Suppose, to the contrary, that A contains at least two triangles, at least one of them having three vertices of attachment. For any such graph G, denote by $d_T(G)$ the minimum distance between two triangles in G such that at least one of them has three vertices of attachment. Construct a graph G by taking a copy of A and by replacing every its triangle $T = \langle\{a,b,c\}\rangle_A$ with three vertices of attachment by a copy of the graph L of Figure 6.

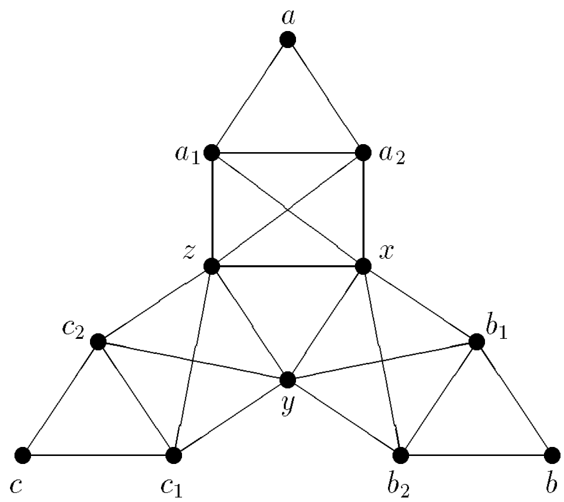


Figure 6

Let $T_1 = \langle\{a,b,c\}\rangle_A$ and T_2 be a pair of triangles in A at minimum distance $d = d_T(A)$ (T_1 having three vertices of attachment). Then the only triangles in the copy of L that can occur in a closed induced subgraph of G as triangles with three vertices of attachment are the triangles $\langle\{x,z,a_1\}\rangle_G$, $\langle\{x,z,a_2\}\rangle_G$, $\langle\{x,y,b_1\}\rangle_G$, $\langle\{x,y,b_2\}\rangle_G$, $\langle\{y,z,c_1\}\rangle_G$, and $\langle\{y,z,c_2\}\rangle_G$. Since the distance of each of them from T_2 is at least $d+1$ (and the pair T_1, T_2 is arbitrary), we have $d_T(G) \geq d_T(A)$, implying that the graph G is A-free. It is apparent that G is also C-free, and it is straightforward to check that its closure $\mathrm{cl}(G)$ contains an induced A. This contradiction shows that it remains to consider the following two cases.

<u>Case 1:</u> A contains one triangle with three vertices of attachment and no other triangle.

<u>Case 2:</u> Every triangle in A has exactly one vertex of attachment.

In Case 1, either $A \simeq N_{i,j,k}$ for some $i,j,k \geq 1$ (and we are done), or some two vertices of attachment of the triangle are connected by a path, but then, replacing one edge of the path by the graph L_2 of Fig. 5, we get a contradiction with the stability. In Case 2, by the claw-freeness, A contains at most two triangles. If A has one triangle, $A \simeq Z_i$ for some $i \geq 1$, and if A has two triangles with a common vertex, then $A \simeq H$. It remains to consider the case when A contains two triangles at a certain nonzero distance (i.e., connected by a path), but then, replacing an arbitrary edge of the path by the graph L_2 of Fig. 5, we get a CA-free graph G such that $\mathrm{cl}(G)$ contains an induced A. This contradiction completes the proof of Theorem 4. ■

3 Hamiltonian results

Denote by $\mathcal{F}_1, \ldots, \mathcal{F}_6$ the classes of graphs shown in Fig. 7 (where the elliptical parts represent cliques of size at least three).

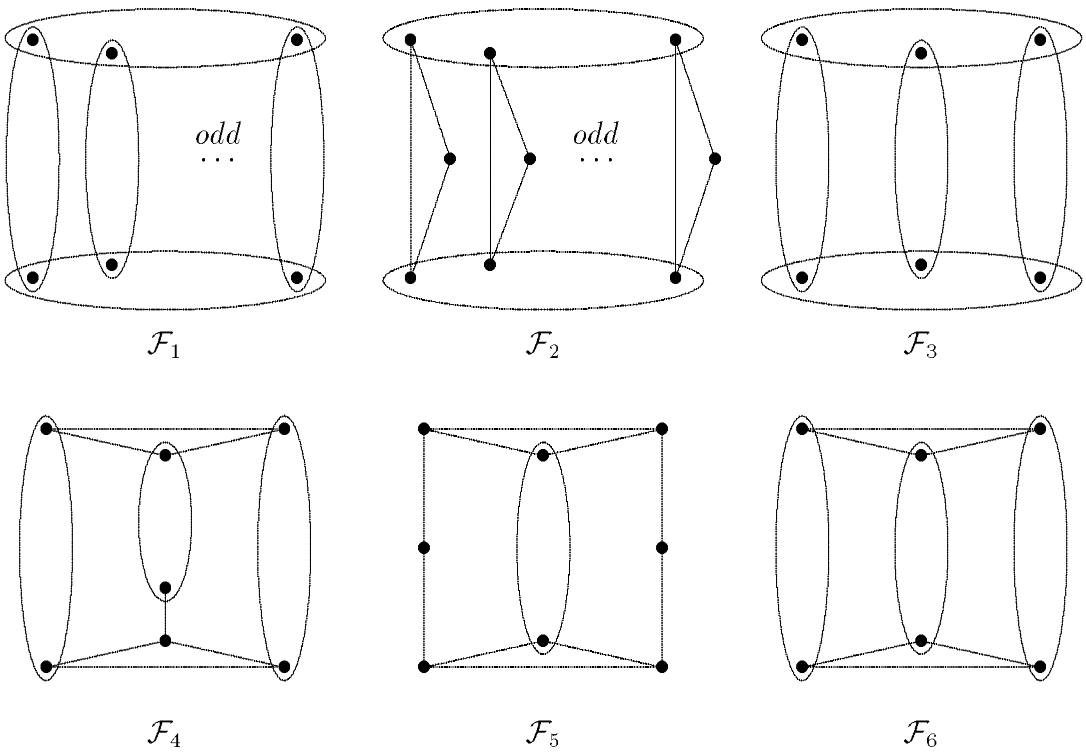
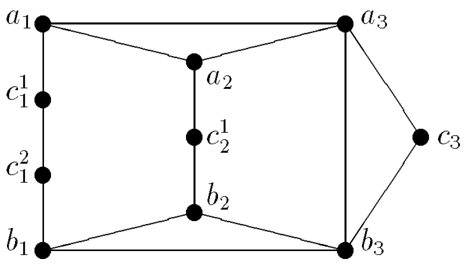


Figure 7

Following [4], further denote by $\mathcal{P}$ the class of all graphs that are obtained by taking two vertex-disjoint triangles $\langle\{a_1, a_2, a_3\}\rangle$, $\langle\{b_1, b_2, b_3\}\rangle$ and by joining every pair of vertices $\{a_i, b_i\}$ by a copy of a path $P_{k_i} = a_i c_i^1 c_i^2 \ldots c_i^{k_i-2} b_i$ for $k_i \geq 3$ or by a triangle $\langle\{a_i, b_i, c_i\}\rangle$. We denote a graph from $\mathcal{P}$ by P_{x_1,x_2,x_3}, where $x_i = k_i$ if a_i, b_i are joined by a copy of P_{k_i}, and $x_i = T$, if a_i, b_i are joined by a triangle (see Fig. 8).

The following results, extending the results of [3], [6], [7], [8], [9] were proved in [5].



$P_{4,3,T} \in \mathcal{P}$

Figure 8

Theorem E [5]. *Let G be a 2-connected graph.*

(i) If G is CP_7-free, then either G is hamiltonian or $\mathrm{cl}(G) \in \mathcal{F}_1$.

(ii) If G is CZ_4-free, then either G is hamiltonian, or $G \in \{P_{3,T,T}, P_{3,3,T}, P_{3,3,3}, P_{4,T,T}\}$, or $\mathrm{cl}(G) \in \mathcal{F}_2$.

(iii) If G is $CN_{1,2,2}N_{1,1,3}$-free, then either G is hamiltonian, or $G \simeq P_{3,3,3}$, or $\mathrm{cl}(G) \in \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5$.

(iv) If G is CE-free, then either G is hamiltonian or $G \in \mathcal{F}_6$.

Corollary F [5].

(i) Every 3-connected CP_7-free graph is hamiltonian.

(ii) Every 3-connected CZ_4-free graph is hamiltonian.

(iii) Every 3-connected $CN_{1,2,2}N_{1,1,3}$-free graph is hamiltonian.

(iv) Every 3-connected CE-free graph is hamiltonian.

Theorem 4 gives a motivation to consider similar questions in the class of CH-free graphs, too. However, it is easy to observe that e.g. the graph obtained by replacing every vertex of an arbitrary cubic 2-connected nonhamiltonian graph G by a triangle (also called the *inflation* of G) is a closed 2-connected nonhamiltonian CH-free graph and hence similar results to Theorem E cannot be expected in this class. Nevertheless, we show that meaningful results can be obtained in classes of graphs defined in terms of triples of forbidden subgraphs, one of them being the hourglass.

We prove the following theorems, in which we denote by $\mathcal{F}_7$, $\mathcal{F}_8$ and $\mathcal{F}_9$ the classes of graphs shown in Figure 9.

Theorem 5. *Let G be a 2-connected CHP_8-free graph. Then either G is hamiltonian or* $\mathrm{cl}(G) \in \mathcal{F}_7$.

Theorem 6. *Let G be a 2-connected CHZ_5-free graph. Then either G is hamiltonian or $G \simeq P_{4,3,3}$ or $\mathrm{cl}(G) \in \mathcal{F}_7$.*

Theorem 7. *Let G be a 2-connected $CHN_{1,1,4}$-free graph. Then either G is hamiltonian or $\mathrm{cl}(G) \in \mathcal{F}_8 \cup \mathcal{F}_9$.*

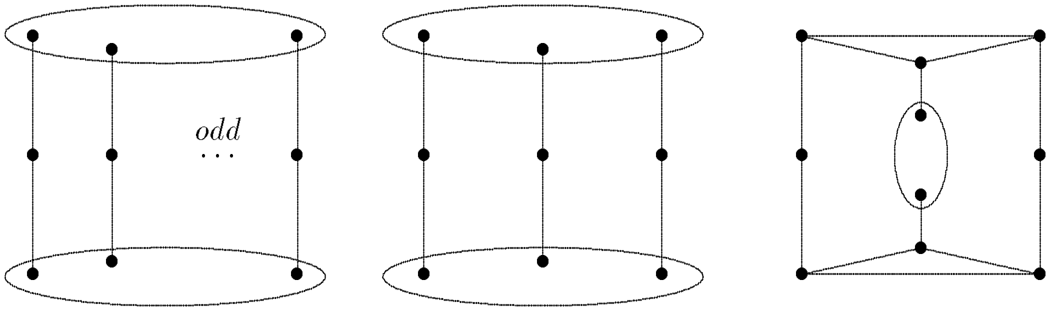


Figure 9

Since every graph in the classes $\mathcal{F}_7$, $\mathcal{F}_8$, $\mathcal{F}_9$ is of connectivity 2, we obtain the following corollary.

Corollary 8.
(i) *Every 3-connected CHP_8-free graph is hamiltonian.*
(ii) *Every 3-connected CHZ_5-free graph is hamiltonian.*
(iii) *Every 3-connected $CHN_{1,1,4}$-free graph is hamiltonian.*

Before proving Theorems 5, 6 and 7, we first introduce some additional notation. The following theorem was proved in [4].

Theorem G [4]. *Every nonhamiltonian 2-connected claw-free graph contains an induced subgraph $F \in \mathcal{P}$.*

Let G be a closed 2-connected nonhamiltonian claw-free graph and let (by Theorem G) $F = P_{x_1,x_2,x_3} \in \mathcal{P}$ be an induced subgraph of G. Recall that we keep the notation of vertices a_i, b_i, c_i^j as in Fig. 8. We denote by:

- K_a the largest clique in G containing the triangle $\langle\{a_1, a_2, a_3\}\rangle_G$,
- K_b the largest clique in G containing the triangle $\langle\{b_1, b_2, b_3\}\rangle_G$,

- for every $i \in \{1,2,3\}$ for which $x_i = T$, by K_i the largest clique in G containing the triangle $\langle\{a_i, c_i, b_i\}\rangle_G$,
- for every $i \in \{1,2,3\}$ for which $x_i \neq T$, by K_i the path $a_i c_i^1 c_i^2 \dots c_i^{k_i-2} b_i$ and by K_i^j $(j = 1, \dots, k_i - 1)$ the largest clique in G containing the j-th edge of the path K_i,
- for every $i \in \{1,2,3\}$, $K_i^* = K_i$ if $x_i = T$, and $K_i^* = \langle \cup_{j=1}^{k_i-1} V(K_i^j)\rangle_G$, if $x_i \neq T$,
- $F^* = \langle V(K_a) \cup V(K_b) \cup (\cup_{i=1}^3 V(K_i^*))\rangle_G$.

Note that since G is closed, all these sets are well-defined.

The following lemma summarizes basic properties of F^*.

Lemma H [5]. *Let G be a closed 2-connected nonhamiltonian claw-free graph and let $F \in \mathcal{P}$ be an induced subgraph of G. Then the graph F^* has the following properties.*

(i) $|V(A_1) \cap V(A_2)| \leq 1$ *for every* $A_1, A_2 \in \{K_a, K_b\} \cup \{K_i |\ x_i = T\} \cup \{K_i^j |\ x_i \neq T, 1 \leq j \leq x_i - 1\}, A_1 \neq A_2$,

(ii) if $x_i = T$ *for some* $i \in \{1,2,3\}$, *then* $V(K_i) \cap V(A) = \emptyset$ *for every* $A \in \{K_j |\ x_j = T\} \cup \{K_j^1 |\ x_j \neq T\} \cup \{K_j^{x_j-1} |\ x_j \neq T\}$, $A \neq K_i$,

(iii) if $x_i \neq T$ *for some* $i \in \{1,2,3\}$, *then* $V(K_i^1) \cap V(K_j^1) = \emptyset$ *and* $V(K_i^{x_i-1}) \cap V(K_j^{x_j-1}) = \emptyset$ *for every* $j \in \{1,2,3\}$ *such that* $j \neq i$ *and* $x_j \neq T$,

(iv) if $x_i = T$ *for at least one* $i \in \{1,2,3\}$, *then* $V(K_a) \cap V(K_b) = \emptyset$,

(v) $a_i, b_i, c_i^\ell \in V_{LD}(G)$ *for* $1 \leq \ell \leq k_i - 2$ *and* $i = 1,2,3$,

(vi) $N_G(a_i) \subset V(K_a) \cup V(K_i^*)$, $N_G(b_i) \subset V(K_b) \cup V(K_i^*)$, $N_G(c_i^\ell) = V(K_i^\ell) \cup V(K_i^{\ell+1})$ *for* $1 \leq \ell \leq k_i - 2$ *and* $i = 1,2,3$.

If G is, moreover, H-free, then we can obtain more information about the structure of the graph F^*.

Lemma 9. *Let G be a closed 2-connected nonhamiltonian CH-free graph and let $F \in \mathcal{P}$ be an induced subgraph of G. Then the graph F^* has the following properties.*

(i) $a_i b_i \notin E(G)$ *for* $i = 1,2,3$,

(ii) $|V(K_i^j)| = 2$ *for every* $i \in \{1,2,3\}$ *and* $j \in \{1, k-1\}$,

(iii) $V(K_a) \cap V(K_b) = \emptyset$.

Proof. Properties (i), (ii) follow immediately from the fact that G is H-free. If $V(K_a) \cap V(K_b) \neq \emptyset$, then, for any $x \in V(K_a) \cap V(K_b)$, $\langle\{x, a_1, a_2, b_1, b_2\}\rangle_G \simeq H$ (by (i) and since F is an induced subgraph). Hence $V(K_a) \cap V(K_b) = \emptyset$. ■

Proof of Theorem 5. By Theorem A and Corollary C(i), it is sufficient to prove that if G is a closed 2-connected CHP_8-free graph, then $G \in \mathcal{F}_7$. Let G be closed, 2-connected and nonhamiltonian. By Theorem G, G contains an induced subgraph $F \in \mathcal{P}$. It is straightforward to check that the only CHP_8-free graph in the class P is the graph $P_{3,3,3}$; hence $F \simeq P_{3,3,3}$. By Lemma 9(ii), $K_i^* \simeq P_3$, $i = 1, 2, 3$. By the claw-freeness and since $a_ib_i \notin E(G)$, $N_G(c_i) = \{a_i, b_i\}$, $i = 1, 2, 3$. By Lemma H(vi), the only vertices of F^* that can have a neighbor in $V(G) \setminus V(F^*)$ are those in $(V(K_a) \cup V(K_b)) \setminus \{a_1, a_2, a_3, b_1, b_2, b_3\}$.

Suppose that $V(G) \neq V(F^*)$ and choose a vertex $u \in V(G) \setminus V(F^*)$ having a neighbor $v \in V(F^*)$. By symmetry, we can suppose that $v \in V(K_a) \setminus \{a_1, a_2, a_3\}$. Since G is closed, $v \in V_{LD}(G)$. If u is adjacent to another vertex $d \in V(G) \setminus V(F^*)$, then $dv \notin E(G)$ (otherwise, since G is closed, $\langle\{v, d, u, a_1, a_2\}\rangle_G \simeq H$). But then $\langle\{d, u, v, a_1, c_1, b_1, b_2, c_2\}\rangle_G \simeq P_8$ – a contradiction. Hence u has no neighbors outside F^*. Since G is closed and $u \notin V(K_a)$, u has no neighbors in $V(K_a)$. Since G is closed and 2-connected, there is a vertex $w \in V(K_b) \setminus \{b_1, b_2, b_3\}$ such that $N_G(u) = \{v, w\}$. Note that, since G is H-free, $vw \notin E(G)$.

Now, since u was arbitrary, repeating this argument we obtain vertices $u_1, \ldots, u_k$, $v_1, \ldots, v_k$ and $w_1, \ldots, w_k$ such that $v_j \in V(K_a) \setminus \{a_1, a_2, a_3\}$, $w_j \in V(K_b) \setminus \{b_1, b_2, b_3\}$, $u_j \in V(G) \setminus V(F^*)$, $N_G(u_j) = \{v_j, w_j\}$ and $v_jw_j \notin E(G)$, $j = 1, \ldots, k$, and such that $V(G) = V(F^*) \cup \{u_1, \ldots, u_k\}$. By the claw-freeness and by the above considerations, all these vertices are distinct. Now it is straightforward to check that k is odd and that $E(G) = E(F^*) \cup \{u_jv_j,\ u_jw_j |\ j = 1, \ldots, k\}$ (otherwise G is hamiltonian). This implies that $G \in \mathcal{F}_7$. ■

Proof of Theorem 6. By Theorems A and D, we can again suppose that G is a closed 2-connected nonhamiltonian CHZ_5-free graph. We show that then either $G \simeq P_{4,3,3}$ or $G \in \mathcal{F}_7$. By Theorem G, the graph G contains an induced subgraph $F \in \mathcal{P}$, and since G is CHZ_5-free, it follows that $F \simeq P_{3,3,3}$ or $F \simeq P_{4,3,3}$. We now consider these two cases separately.

<u>Case 1:</u> $F \simeq P_{4,3,3}$. First observe that $|V(K_a)| = |V(K_b)| = 3$, since if e.g. $d \in V(K_a) \setminus \{a_1, a_2, a_3\}$, then $\langle\{a_1, d, a_2, c_1^1, c_1^2, b_1, b_3, c_3\}\rangle_G \simeq Z_5$. Secondly, since G is H-free and, by Lemma 9(ii), we have $|V(K_1^1)| = |V(K_1^3)| = |V(K_2^1)| = |V(K_2^2)| = |V(K_3^1)| = |V(K_3^2)| = 2$, implying $N_G(c_i) = \{a_i, b_i\}$ for $i = 2, 3$. If $|V(K_2^2)| \geq 3$, then, for a vertex $d \in V(K_2^2) \setminus \{c_1^1, c_1^2\}$, we have $\langle\{c_1^2, d, c_1^1, b_1, b_2, c_2, a_2, a_3\}\rangle_G \simeq Z_5$. Hence also $|V(K_2^2)| = 2$, implying that $N_G(c_1^1) = \{a_1, c_1^2\}$ and $N_G(c_1^2) = \{c_1^1, b_1\}$. But then no vertex of F can have a neighbor outside F and, since G is connected, $G \simeq F \simeq P_{4,3,3}$.

<u>Case 2:</u> $F \simeq P_{3,3,3}$. Similarly as above, $|V(K_i^j)| = 2$ for $i = 1, 2, 3$, $j = 1, 2$, and hence $N_G(c_i) = \{a_i, b_i\}$, $i = 1, 2, 3$.

Let $u \in V(G) \setminus V(F^*)$ be a vertex having a neighbor v in $V(F^*)$. By symmetry and by Lemma H(vi), we can suppose that $v \in V(K_a) \setminus \{a_1, a_2, a_3\}$. If u has another neighbor $d \in V(G) \setminus V(F^*)$, then, since (by the H-freeness) $vd \notin E(G)$, $\langle\{b_1, b_2, b_3, c_1, a_1, v, u, d\}\rangle_G \simeq Z_5$. Hence u has no neighbors outside F^*. It is apparent that u has no other neighbors in K_a except v, and thus, since G is closed and 2-connected, there is a vertex $w \in V(K_b) \setminus \{b_1, b_2, b_3\}$ such that $N_G(u) = \{v, w\}$. Since G is H-free, $vw \notin E(G)$. But then we are in the same situation as in final part of the proof of Theorem 5 and, using the same arguments, we obtain $G \in \mathcal{F}_7$.

Proof of Theorem 7. Let G be a closed (cf. Theorem A and Corollary C(ii)) 2-connected nonhamiltonian $CHN_{1,1,4}$-free graph. We show that $G \in \mathcal{F}_8 \cup \mathcal{F}_9$.

By Theorem G, the graph G contains an induced subgraph $F \in \mathcal{P}$, and since G is $CHN_{1,1,4}$-free, we have $F \simeq P_{3,3,3}$ or $F \simeq P_{4,3,3}$. We again consider these cases separately.

<u>Case 1:</u> $F \simeq P_{3,3,3}$. Similarly to the previous proofs, $|V(K_i^j)| = 2$ for $i = 1, 2, 3$, $j = 1, 2$, and thus $N_G(c_i) = \{a_i, b_i\}$, $i = 1, 2, 3$. By Lemma H(vi), the only vertices having neighbors outside F^* can be those in $(V(K_a) \cup V(K_b)) \setminus \{a_1, a_2, a_3, b_1, b_2, b_3\}$. If e.g. $v \in V(K_a) \setminus \{a_1, a_2, a_3\}$ has a neighbor $u \in V(G) \setminus V(F^*)$, then $\langle\{b_1, b_2, b_3, c_2, c_3, c_1, a_1, v, u\}\rangle_G \simeq N_{1,1,4}$ – a contradiction. Thus, by symmetry, $V(G) = V(F^*)$. Since G is closed and nonhamiltonian, we have also $E(G) = E(F^*)$ and hence $G \in \mathcal{F}_8$.

<u>Case 2:</u> $F \simeq P_{4,3,3}$. First observe that, like in Case 1, $|V(K_1^1)| = |V(K_1^3)| = 2$ and $|V(K_i^j)| = 2$ for $i = 2, 3$, $j = 1, 2$, implying that $N_G(c_i) = \{a_i, b_i\}$, $i = 2, 3$. We show that $|V(K_a)| = |V(K_b)| = 3$. Let, to the contrary, $d \in V(K_a) \setminus \{a_1, a_2, a_3\}$. Then, by Lemma 9($iii$), $d \notin V(K_b)$ and thus $\langle\{b_1, b_2, b_3, c_2, c_3, c_1^2, c_1^1, a_1, d\}\rangle_G \simeq N_{1,1,4}$. Hence $|V(K_a)| = |V(K_b)| = 3$, implying that the only vertices of F^* that can have neighbors outside F^* are those in K_2^2.

Let $u \in V(G) \setminus V(F^*)$ be adjacent to $v \in V(K_1^2)$. By Lemma H(vi), $v \in V(K_1^2) \setminus \{c_1^1, c_1^2\}$. Since G is closed, u has no other neighbors (except v) in K_1^2. Since G is 2-connected, there is a path P in G starting at u and ending at a vertex $w \in V(K_1^2) \setminus \{v, c_1^1, c_1^2\}$ with all interior vertices outside F^*. Let z be the first vertex of P different from u. Since G is H-free, we have $zv \notin E(G)$, but then $\langle\{b_1, b_2, b_3, c_2, c_3, c_1^2, v, u, z\}\rangle_G \simeq N_{1,1,4}$. This contradiction shows that no vertex in F^* can have a neighbor outside F^*, i.e., $V(G) = V(F^*)$. It is straightforward to check that also $E(F^*) = E(G)$ (otherwise G is hamiltonian), which implies that $G \in \mathcal{F}_9$. ■

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