Closure concepts – a survey

Hajo Broersma
Faculty of Mathematical Sciences
University of Twente
P.O. Box 217
7500 AE Enschede
The Netherlands
e-mail broersma@math.utwente.nl

Zdeněk Ryjáček *
Department of Mathematics
University of West Bohemia
P.O. Box 314
306 14 Plzeň
Czech Republic
e-mail ryjacek@kma.zcu.cz

Ingo Schiermeyer
Department of Mathematics
Technical University of Freiberg
D-09596 Freiberg
Germany
e-mail schierme@math.tu-freiberg.de

September 23, 1999

Abstract

In this paper we survey results of the following type (known as *closure results*). Let P be a graph property, and let $C(u,v)$ be a condition on two nonadjacent vertices u and v of a graph G. Then $G+uv$ has property P if and only if G has property P. The first and now well-known result of this type was established by Bondy and Chvátal in a paper published in 1976: If u and v are two nonadjacent vertices with degree sum n in a graph G on n vertices, then $G+uv$ is hamiltonian if and only if G is hamiltonian. Based on this result, they defined the n-closure $\mathrm{cl}_n(G)$ of a graph G on n vertices as the graph obtained from G by recursively joining pairs of nonadjacent vertices with degree sum n until no such pair remains. They showed that $\mathrm{cl}_n(G)$ is well-defined, and that G is hamiltonian if and only if $\mathrm{cl}_n(G)$ is hamiltonian. Moreover, they showed that $\mathrm{cl}_n(G)$ can be obtained by a polynomial algorithm, and that a Hamilton cycle in $\mathrm{cl}_n(G)$ can be transformed into a Hamilton cycle of G by a polynomial algorithm. As a consequence, for any graph G with $\mathrm{cl}_n(G) = K_n$ (and $n \geq 3$), a Hamilton

*Research supported by grant GA ČR No. 201/97/0407

cycle can be found in polynomial time, whereas this problem is NP-hard for general graphs. All classic sufficient degree conditions for hamiltonicity imply a complete n-closure, so the closure result yields a common generalization as well as an easy proof for these conditions. In their first paper on closures, Bondy and Chvátal gave similar closure results based on degree sum conditions for nonadjacent vertices for other graph properties. Inspired by their first results, many authors developed other closure concepts for a variety of graph properties, or used closure techniques as a tool for obtaining deeper sufficiency results with respect to these properties. Our aim is to survey this progress on closures made in the past (more than) twenty years.

1 Introduction

We use [10] for terminology and notation not defined here and consider simple graphs only.

In the sequel let $G = (V, E)$ be a 2-connected graph on n vertices with vertex set V and edge set E. Let $N_G(x)$ denote the set of neighbors of the vertex $x \in V$, and let $d_G(x) = |N_G(x)|$ denote the degree of x. Let $N_G[x] = N_G(x) \cup \{x\}$; if no ambiguity can arise, we omit the subscripts.

More than 20 years ago, the now well-known paper 'A method in graph theory' by Bondy and Chvátal [9] was published. The closure concept introduced in this paper opened a new horizon for the research on hamiltonian and related properties of graphs. Closure concepts now play an important role in results on the existence of cycles, paths, and other subgraphs in graphs. They generalize the classic degree conditions, and yield polynomial algorithms for problems that are NP-hard in general. But they also have proved to be a useful tool in proving deeper results. We focus on the hamiltonian problem first, and mention the main results in [9]. Some of the proofs will be outlined in the next section.

A graph G is *hamiltonian* if it contains a *Hamilton cycle*, i.e. a cycle containing all the vertices of G. A *Hamilton path* of G is a path containing all the vertices of G. As remarked in [25], the following method was found in an attempt to find a constructive proof for a sufficient condition for hamiltonicity based on degree sequences. It exploits the following result due to Ore [60].

Theorem 1.1 [60]. *Let u and v be distinct nonadjacent vertices of a graph G such that $d(u) + d(v) \geq n$. Then G is hamiltonian if and only if $G + uv$ is hamiltonian.*

Bondy and Chvátal defined the *n-closure* $\mathrm{cl}_n(G)$ as the graph obtained from G by recursively joining pairs of nonadjacent vertices the degree sum of which is at least n until no such pair remains. They showed that $\mathrm{cl}_n(G)$ is well-defined, i.e. is uniquely determined by G, and that G is hamiltonian if and only if $\mathrm{cl}_n(G)$ is hamiltonian.

Adding edges to a graph makes it intuitively more likely that the new graph contains a Hamilton cycle. For the n-closure of a graph this is not the case: the n-closure has a

Hamilton cycle if and only if the original graph has one. But it could be more likely that we can find a Hamilton cycle in the n-closure more easily. Note that, e.g., in a complete graph on at least three vertices finding a Hamilton cycle is a trivial exercise.

As a consequence of Theorem 1.1, if G is a graph on at least three vertices and $\mathrm{cl}_n(G)$ is a complete graph, then G is hamiltonian. The authors showed in [9] that $\mathrm{cl}_n(G)$ can be constructed by a polynomial algorithm. Moreover, they showed that if $\mathrm{cl}_n(G)$ is complete, then any Hamilton cycle in K_n can be transformed into a Hamilton cycle in G in polynomial time. These two results together imply that finding a Hamilton cycle in a graph G for which $\mathrm{cl}_n(G) = K_n$ is a polynomial problem, whereas this problem is NP-hard in general.

It was noted in [9] that many of the classic sufficient degree conditions for hamiltonicity guarantee that $\mathrm{cl}_n(G) = K_n$, yielding easier proofs of the corresponding results. But apart from generalizing degree conditions and yielding polynomial algorithms, closure results also have proved to be useful as a tool in proving deeper sufficiency results, because they give more information on the structure of the (closure of the) graphs under consideration. We postpone the details to the next section.

A drawback of the above closure method is that $\mathrm{cl}_n(G)$ can only be complete if G is very dense, i.e. $|E| \geq \lfloor \frac{(n+2)^2}{8} \rfloor$ (see [26]). Many of the later closure concepts have the same disadvantage, but some of them can also be applied to graphs with a number of edges which is a linear function of n. For the latter concepts we often lose the nice property of uniqueness of the associated closure.

For many other graph properties P, Bondy and Chvátal [9] have found sufficient conditions of the type $d(u) + d(v) \geq k$ (for the smallest possible value of k) such that if $G + uv$ has property P, then G itself has property P. Such properties are called *k-stable*, and the k-closure is defined by recursively joining pairs of nonadjacent vertices with degree sum at least k until no such pair remains. More details can be found in the next section.

Inspired by the above results several other closure concepts have been developed. Most of these concepts are based on conditions comparing the degree sum of a pair of nonadjacent vertices with some global parameter of the graph, e.g., (a function of) the number of vertices of the graph or some subgraph. These results will be discussed in Section 2. Some of the more recent closure concepts try to take into account the local structure of the graph close to the nonadjacent pair under consideration. These results have been gathered in Sections 3 and 4, where the latter one is particularly devoted to closure concepts in claw-free graphs.

2 Degree conditions

2.1 k-closure and k-stability

In [9], Bondy and Chvátal introduced the closure of a graph and the stability of a property. The *k-closure* $\mathrm{cl}_k(G)$ of a graph G is obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least k, until no such pair remains.

They showed that $\mathrm{cl}_k(G)$ is unique, i.e. it is independent of the order of addition of the edges. It is clear that any graph G of order n satisfies

$$G = \mathrm{cl}_{2n-3}(G) \subseteq \mathrm{cl}_{2n-4}(G) \subseteq \cdots \subseteq cl_1(G) \subseteq \mathrm{cl}_0(G) = K_n,$$

where $H \subseteq G$ denotes that H is a spanning subgraph of G.

A property P defined on all graphs of order n is said to be *k-stable* if for any graph of order n that does not satisfy P, the fact that uv is not an edge of G and that $G + uv$ satisfies P implies $d_G(u) + d_G(v) < k$. Vice versa, if $uv \notin E(G)$, $d_G(u) + d_G(v) \geq k$ and $G + uv$ has property P, then G itself has property P. Every property is $(2n - 3)$-stable and every k-stable property is $(k + 1)$-stable. We denote by $s(P)$ the smallest integer k such that P is k-stable and call it the *stability* of P. This number usually depends on n and is at most $2n - 3$.

The most well-known application of this closure method is the property "hamiltonicity" (cf. Theorem 1.1 in the introduction). The basic idea behind it can be described as follows. Suppose that a Hamilton cycle C in $G + uv$ contains the edge uv and that the vertices of C are labeled $u = v_1, v_2, \ldots, v_n = v$ according to their cyclic ordering around C. Since $d_G(u) + d_G(v) \geq n$, there exists an integer i with $2 \leq i \leq n-2$ such that $uv_{i+1}, vv_i \in E(G)$. Now $C' = v_1 v_{i+1} v_{i+2} \ldots v_n v_i v_{i-1} \ldots v_1$ is a Hamilton cycle in G. The transformation from C in $G+uv$ to C' in G using the edges uv_{i+1} and vv_i is referred to as a *standard transformation*, sometimes called *crossing*.

Bondy and Chvátal showed in [9] that $\mathrm{cl}_k(G)$ can be constructed in polynomial time $O(n^4)$. In [76], Szwarcfiter presented an algorithm that computes $\mathrm{cl}_k(G)$ in time $O(n^3)$. Another algorithm that computes $\mathrm{cl}_k(G)$ in time $O(n^3)$ is due to Khuller [50]. Parallel algorithms for the computation of $\mathrm{cl}_k(G)$ can be found in [71]. The computational (parallel) complexity for the computation of $\mathrm{cl}_k(G)$ has been resolved by Monti [57].

Moreover, Bondy and Chvátal showed that if $\mathrm{cl}_n(G)$ is complete, then any Hamilton cycle in K_n can be transformed into a Hamilton cycle in G in polynomial time $O(n^3)$.

It was shown in [26] that $\mathrm{cl}_k(G)$ can only be complete if $|E(G)| \geq \lfloor \frac{(k+2)^2}{8} \rfloor = f(k)$. Moreover, for all n and k with $0 \leq k \leq 2n - 3$ they constructed a graph G with $f(k)$ edges and a complete k-closure.

2.2 Closure extensions for hamiltonicity

Several extensions of the closure concept for hamiltonicity have been established. Preserving uniqueness, the following two concepts are of major importance.

2.2.1 The 0-dual closure

In [2], Ainouche and Christofides introduced the concept of the 0-*dual closure* of a graph.

Theorem 2.1 [2]. *Let u and v be two nonadjacent vertices of a 2-connected graph G, let $T := \{w \in V(G) \setminus \{u,v\} | u,v \notin N(w)\}$, $t := |T|$, $\lambda_{uv} := |N(u) \cap N(v)|$, and let $d_1^T \le d_2^T \le \ldots \le d_t^T$ be the degree sequence of the vertices of T (in G). If*

$$d_i^T \ge t+2 \text{ for all } i \text{ with} \max(1, \lambda_{uv}-1) \le i \le t, \tag{1}$$

then G is hamiltonian if and only if $G+uv$ is hamiltonian.

In [2], the corresponding (unique) closure of G is called the 0-*dual closure* and denoted by $C_0^*(G)$. Since Theorem 2.1 is more general than Theorem 1.1, $G \subseteq \mathrm{cl}_n(G) \subseteq C_0^*(G)$.

Corollary 2.2 [2]. *Let G be a 2-connected graph. If $C_0^*(G)$ is complete, then G is hamiltonian.*

In [18], it was observed that (1) can be restated in terms of degree sums of independent triples.

Proposition 2.3 [18]. *Statement (1) is equivalent to*

$$d(u)+d(v)+d(w) \ge n+\lambda_{uv} \text{ for at least } \min(t, t+2-\lambda_{uv}) \text{ vertices } w \in T. \tag{2}$$

A slightly weaker condition than the one of Theorem 2.1 was obtained by Zhu, Tian and Deng [79].

2.2.2 The triple closure

In [18], Broersma and Schiermeyer introduced the concept of the *triple closure* of a graph.

First we introduce some additional notation. For a vertex $w \in T$, we let $\eta(w) = |N(w) \setminus T|$, and we let $\eta_1 \ge \eta_2 \ge \ldots \ge \eta_t$ denote the ordered sequence corresponding to the set $\{\eta(w)|w \in T\}$. We say that G satisfies the 1-2-3-condition if $T = \emptyset$ or $\eta_i \ge 4-i$ for all i with $1 \le i \le t$ (Note that $t \ge 1$ implies $\eta_1 \ge 3$, $t \ge 2$ implies $\eta_2 \ge 2$, and $t \ge 3$ implies $\eta_3 \ge 1$).

Theorem 2.4 [18]. *Let u and v be two nonadjacent vertices of a 2-connected graph G of order n. If $\lambda_{uv} \ge 3$ and*

$$|N(u) \cup N(v) \cup N(w)| \ge n - \lambda_{uv} \text{ for at least } t+2-\lambda_{uv} \text{ vertices } w \in T, \tag{3}$$

or if $\lambda_{uv} \le 2$ and G satisfies the 1-2-3-condition and

$$|N(u) \cup N(v) \cup N(w)| = n-3 \text{ for all vertices } w \in T, \tag{4}$$

then G is hamiltonian if and only if $G+uv$ is hamiltonian.

Successively joining pairs of nonadjacent vertices u and v satisfying the conditions of Theorem 2.4 as long as this is possible (in the new graphs) a unique graph is obtained from G. In [18] this graph is called the *triple closure* of G and denoted by $TC(G)$.

Moreover, in [18] it is shown that $G \subseteq \mathrm{cl}_n(G) \subseteq TC(G)$ for any graph G.

Let p, q, r be three integers such that $p, q, r \geq 3$ and $p + q + r = n$. Let G_{pqr} denote the graph on n vertices obtained from three disjoint complete graphs $H_1 = K_p, H_2 = K_q$ and $H_3 = K_r$ by adding the edges of two triangles between two disjoint triples of vertices, each containing one vertex of each of H_1, H_2 and H_3. Moreover, let G^+_{pqr} denote the graph obtained from G_{pqr} by adding an edge joining a vertex of H_1 and one of H_2, both not incident with edges of the added triangles.

It is easy to check that G_{pqr} is nonhamiltonian, and that the addition of any new edge to G_{pqr} yields a hamiltonian graph. As noted in [18], graphs G^+_{pqr} have a complete triple closure, i.e. $TC(G^+_{pqr}) = K_{p+q+r}$, while, if $p, q \geq 4$, $\mathrm{cl}_n(G) = C^*_0(G) = G$.

The graphs G_{pqr} show that we cannot omit the 1-2-3-condition in Theorem 2.4.

2.2.3 Further closure concepts

In [35], Faudree et al. defined the $(n-2)$-*neighborhood closure* of a 2-connected graph G, denoted by $N_{n-2}(G)$, as the (unique) graph obtained from G by successively joining pairs of nonadjacent vertices u and v satisfying $|N(u) \cup N(v)| \geq n - 2$. Since for such pairs $T_{uv} = \emptyset$, it is clear that $N_{n-2}(G) \subseteq C^*_0(G)$ and $N_{n-2}(G) \subseteq TC(G)$ for any graph G.

In [1], Ainouche and Christofides consider the cardinality of a maximum independent set containing two nonadjacent vertices u, v, which is denoted by α_{uv}.

Theorem 2.5 [1]. *Let u, v be two nonadjacent vertices of a 2-connected graph G such that*

$$\alpha_{uv} \leq \lambda_{uv}.$$

Then G is hamiltonian.

Theorem 2.5 also generalizes Theorem 1.1. However, testing whether a given pair of nonadjacent vertices u, v satisfies the condition may take exponential time.

Theorem 2.5 has been successfully applied in [27] for establishing a sufficient condition for hamiltonian graphs.

2.3 Closures of graph classes

In this section the closures for several classes of hamiltonian graphs will be determined. Let

$$\sigma_k := \min\{\sum_{i=1}^{k} d(v_i) | \{v_1, v_2, \ldots, v_k\} \text{ is an independent set of vertices in } G\}$$

if $\alpha(G) \geq k$, and $\sigma_k = \infty$ otherwise $(k \geq 2)$.

We first state a number of known sufficient degree conditions for hamiltonicity.

Theorem 2.6 (Dirac, 1952) [28]. *Let G be a graph of order $n \geq 3$ with $\sigma_1(G) = \delta(G) \geq n/2$. Then G is hamiltonian.*

Theorem 2.7 (Ore, 1960) [60]. *Let G be a graph of order $n \geq 3$ with $\sigma_2(G) \geq n$. Then G is hamiltonian.*

Let $d_1 \leq d_2 \leq \ldots \leq d_n$ denote the *degree sequence* of a graph G on n vertices.

Theorem 2.8 (Pósa, 1962) [62]. *If $d_k > k$ for $1 \leq k \leq (n-1)/2$ and $d_{(n+1)/2} \geq (n+1)/2$, if n is odd, then G is hamiltonian.*

Theorem 2.9 (Bondy, 1969) [8]. *If $d_j + d_k \geq n$ for all pairs j, k with $j < k, d_j \leq j, d_k \leq k-1$, then G is hamiltonian.*

Theorem 2.10 (Chvátal, 1972) [24]. *If $d_{n-k} \geq n-k$ for all k with $d_k \leq k < n/2$, then G is hamiltonian.*

Theorem 2.11 (Las Vergnas, 1972) [53]. *If there exists a labeling $v_1, v_2, \ldots, v_n$ of the vertices such that $j < k, k \geq n-j$, $v_j v_k \notin E(G)$, $d(v_j) \leq j$, and $d(v_{k-1}) \leq k-1$ implies $d(v_j) + d(v_k) \geq n$, then G is hamiltonian.*

Each of the above theorems is more general than its preceding one. Moreover, all hamiltonian graphs detected by these six theorems have a complete n-closure $\mathrm{cl}_n(G)$ (cf. [54]).

The following two theorems extend Theorem 2.10.

Theorem 2.12 (Zhu and Tian, 1983) [78]. *Let G be a graph of order $n \geq 3$ and denote by $S(m)$ and $S^+(m)$ the sets of all vertices of degree m or $\geq m$, respectively. If each i with $d_i \leq i < n/2$ satisfies one of the following three conditions*

(i) $d_{n-i} \geq n-i$,

(ii) $d_{n-i} = n-i-1, d_{n-i+1} \geq n-i$ *and there exists an integer $r, 1 \leq r \leq i$, such that* $d_{i+r} < n-i-r$ *and* $\sum_{j=1}^{r}(r+1-j)(|S(n-j)| + |S(n-i-j)|) \geq \sum_{j=1}^{r} d_{i+j} + 1$,

(iii) $d_{n-i} = d_{n-i+1} = n-i-1$, *and there exists an integer $r, 1 \leq r \leq i$, such that* $d_{i+r} < n-i-r, d_{i+j} \geq i+j (1 \leq j \leq r)$ *and* $\sum_{j=1}^{r}(r+1-j)(|S(n-j)|+|S(n-i-j)|) \geq \sum_{j=1}^{r} d_{i+j} + r(i - |S^+(n-i)|) + 1$,

then $\mathrm{cl}_n(G)$ is complete (and hence G is hamiltonian).

For two disjoint graphs G and H, let $G + H$ denote the *union* of G and H, and let $G \vee H$ denote the *join* of G and H obtained from $G + H$ by joining all vertices of G to all vertices of H.

Theorem 2.13 (Dirac, 1973) [29]. *Let G be a graph of order $n \geq 3$ with the following properties:*

(i) $d_1 \geq 2$,

(ii) for each integer with $2 \leq i \leq n/2 - 1, d_i \leq i$ *implies* $d_{n-i+1} \geq n-i$,

(iii) $d_{(n+7)/2} \geq (n+1)/2$,

(iv) $G \notin \mathcal{G}$, *where* $\mathcal{G}$ *denotes the class of graphs with* $n \geq 5$ *such that either*

$$G \subseteq K_j \vee (K_{n-2j} + jK_1), \text{ with } 2 \leq j \leq n/2 - 1,$$

or n *is odd and*

$$G \subseteq K_{(n-1)/2} \vee ((n+1)/2)K_1,$$

or

$$G \subseteq K_{(n-3)/2} \vee (2K_2 + ((n-5)/2)K_1),$$

or

$$G \subseteq K_{(n-5)/2} \vee (2K_3 + ((n-7)/2)K_1) \text{ for } n \geq 7.$$

Then G *is hamiltonian.*

For these graphs Broersma [16] has determined their n-closures.

Theorem 2.14 [16]. *Let* G *be a graph of order* $n \geq 3$ *satisfying the hypothesis of Theorem 2.13. Then either* $\mathrm{cl}_n(G)$ *is complete or* $\mathrm{cl}_n(G) \in \{C_5\} \cup \mathcal{G}_1 \cup \mathcal{G}_2$, *where*

$\mathcal{G}_1 = \{K_{r-1} \vee (i_1P_1 + i_2P_2) | r \geq 3, n = 2r+1, i_1, i_2 \geq 0,\ i_1 + 2i_2 = r+2, i_1 + i_2 \leq r-1\}$,

$\mathcal{G}_2 = \{K_{r-2} \vee (\sum_{k=1}^{r+3} i_kP_k + \sum_{k=3}^{r+3} j_kC_k) | r \geq 3, n = 2r+1,\ i_k \geq 0$ *for* $1 \leq k \leq r+3, j \geq 0$ *for* $3 \leq k \leq r+3$, $\sum_{k=1}^{r+3} ki_k + \sum_{k=3}^{r+3} kj_k = r+3, \sum_{k=1}^{r+3} i_k + \sum_{k=3}^{r+3} j_k \leq r-2,\ i_1 + \sum_{k=2}^{r+3} 2i_k \leq r-2\}$.

Theorem 2.7 was extended as follows by several mathematicians. The next two results are due to Hayes and Schmeichel [44] and Schiermeyer [73], respectively.

A graph G is *1-tough* if the number of components of $G - S$ is at most $|S|$ for every cut set $S \subset V(G)$.

Theorem 2.15 [44]. *Let* G *be a 1-tough graph of order* $n \geq 3$ *with* $\sigma_2(G) \geq n-2$. *Then* G *is hamiltonian.*

Theorem 2.16 [73]. *Let* G *be a graph of order* $n \geq 3$ *satisfying the hypothesis of Theorem 2.15. Then* $C_0^*(G)$ *is complete.*

In [74], Skupień has determined the n-closures of all nonhamiltonian graphs G satisfying $\sigma_2(G) \geq n-3$. Jung [49] proved that all 1-tough graphs with $\sigma_2 \geq n-4$ and $n \geq 11$ are hamiltonian.

Faudree et al. [36] proved the following sufficient condition for hamiltonian graphs based on neighborhood unions.

Theorem 2.17 [36]. *Let* G *be a 2-connected graph of order* $n \geq 3$. *If* $|N(u) \cup N(v)| \geq (2n-1)/3$ *for all pairs of nonadjacent vertices* $u, v \in V(G)$, *then* G *is hamiltonian.*

Computing the 0-dual closure Schiermeyer [73] obtained the following result with a slight improvement.

Theorem 2.18 [73]. *Let* G *be a 2-connected graph of order* $n \geq 3$. *If* $|N(u) \cup N(v)| \geq (2n-2)/3$ *for all pairs of nonadjacent vertices* $u, v \in V(G)$, *then* $C_0^*(G)$ *is complete.*

The following result using both degree sums and connectivity is due to Bauer et al. [4].

Theorem 2.19 [4]. *Let G be a 2-connected graph of order n with $\sigma_3 \geq n + \kappa$. Then G is hamiltonian.*

Schiermeyer [73] established the following more general result.

Theorem 2.20 [73]. *Let G be a graph of order $n \geq 3$ satisfying the hypothesis of Theorem 2.19. Then C_0^* is complete.*

2.4 The complete closure

Closure Theory (so far) is based on the fact that if the property P is k-stable and if $\mathrm{cl}_k(G)$ satisfies P, then G itself satisfies P. But it is not always easier to check a property P in $\mathrm{cl}_k(G)$ than in G, and since the complete graph K_n has a lot of interesting properties, this theory is often used in a weaker form, by proving that $\mathrm{cl}_k(G)$ is complete. This led Faudree et al. [31] to introduce the complete closure of a graph and the complete stability of a property of graphs. In the sequel we give a sketch of a proof for the result on the property “pancyclicity”.

The *complete closure number* $\mathrm{cc}(G)$ of a graph G of order n is the largest integer $k \leq 2n-3$ such that $\mathrm{cl}_k(G) = K_n$. For example, $\mathrm{cc}(K_n) = 2n-3, \mathrm{cc}(K_n - e) = 2n-4, \mathrm{cc}(\overline{K}_n) = 0$, and $\mathrm{cc}(G) = 2d$ if G is d-regular.

The *complete stability* $\mathrm{cs}(P)$ of a property P defined on all graphs of order n and satisfied by K_n is the smallest integer k such that any graph G satisfies P if $\mathrm{cl}_k(G)$ is complete. This number usually depends on n and satisfies $\mathrm{cs}(P) \leq s(P)$.

For the properties that are related to the existence of cycles, paths, or cliques in G, Faudree et al. (cf. [31]) generally obtained better (that is smaller) values for $\mathrm{cs}(P)$ than for $s(P)$.

2.5 Graph properties

In this section we shortly list the stability and complete stability for several graph properties. We will also give limit examples, i.e. examples that show the sharpness of the results (if known) and state some conjectures and open problems.

2.5.1 ”G contains a cycle C_k”

Theorem 2.21 (Bondy and Chvátal) [9]. *The property P: ”G contains a cycle C_k” satisfies $s(P) = 2n-k$ for $5 \leq k \leq n$ if k is odd and $s(P) = 2n-k-1$ for $4 \leq k \leq n$ if k is even.*

For $k = n$ we obtain Theorem 1.1.

Moreover, it was observed by several mathematicians (cf., e.g., [15]) that the property P: "G has circumference k" satisfies $s(P) = n$ also for $k < n$. This was improved by Zhu, Tian and Deng [79] as follows.

Theorem 2.22 [79]. *Let $p = 2n - k - d(u) - d(v)$ and $T' = \{w \in T | d(w) \geq n + t - k + \max\{2, p\}\}$ with $t' = |T'|$. If $d(u) + d(v) \geq 2n - k - t'$, then G contains a cycle C_k if $G + uv$ contains a cycle C_k.*

Theorem 2.23 (Faudree et al.) [31]. *For every integer r between 1 and $\lfloor n/2 \rfloor - 1$, the property P: "G contains a cycle C_{2r+1}" satisfies $\mathrm{cs}(P) = n + 1$ if n is even and $n \leq \mathrm{cs}(P) \leq n + 1$ if n is odd.*

Limit example for n even: $K_{\frac{n}{2},\frac{n}{2}}$.

Theorem 2.24 [69]. *When n is odd the property P: "G contains a cycle C_k" satisfies $\mathrm{cs}(P) \leq n$ for each integer k between 3 and $(n + 13)/5$.*

This improves the previous bound of $(n + 19)/13$ given in [32].

2.5.2 "G is k-hamiltonian or k-edge-hamiltonian"

A graph G is called *k-hamiltonian* or *k-edge-hamiltonian* if the deletion of at most k vertices or pairwise disjoint edges from G results in a hamiltonian graph, respectively.

Theorem 2.25 (Bondy and Chvátal) [9]. *The property P: "G is k-hamiltonian or k-edge-hamiltonian" satisfies $s(P) = n + k$.*

This was improved by Zhu, Tian and Deng [79].

Theorem 2.26 [79]. *Let k, n be positive integers with $k \leq n - 3$. Let $p = n + k - d(u) - d(v)$ and $T' = \{w \in T | d(w) \geq t + k + \max\{2, p\}\}$. If $d(u) + d(v) \geq n + k - t'$, then G is k-hamiltonian or k-edge-hamiltonian if $G + uv$ is k-hamiltonian or k-edge-hamiltonian, respectively.*

Theorem 2.27 (Faudree et al.) [31]. *The property P: "G is k-hamiltonian or k-edge-hamiltonian" satisfies $\mathrm{cs}(P) = n + k$.*

Limit example: $K_{k+1} + (K_{n-k-2} \cup K_1)$.

2.5.3 "G contains two edge-disjoint Hamilton cycles"

Theorem 2.28 (Faudree et al.) [31]. *The property P: "G contains two edge-disjoint Hamilton cycles" satisfies $n + 2 \leq \mathrm{cs}(P) \leq n + 4$.*

2.5.4 "G contains a cycle with vertex v"

In [63] it was observed that the proof of Theorem 2.21 remains valid if we prescribe an arbitrary vertex v to be contained in the k-cycle. They proved the next two results.

Theorem 2.29 [63]. *The property P: "G contains a cycle C_k with vertex v" satisfies $s(P) = 2n - k$ for $5 \leq k \leq n$ and $s(P) = 2n - k - 1$ for $4 \leq k < n$ if k is even.*

Theorem 2.30 [63]. *Let $c > 0$ and $cn \geq 6\lceil \frac{1}{c} \rceil - 2$. If $\mathrm{cl}_{(1+c)n}(G)$ is complete, then every vertex is contained in a cycle C_k for $6\lceil \frac{1}{c} \rceil - 1 \leq k \leq n$.*

2.5.5 "G contains a cycle with edge e"

In [63] it was observed that the proof of Theorem 2.21 remains valid also if we prescribe an arbitrary edge e to be contained in the k-cycle.

Theorem 2.31 [63]. *The property P: "G contains a cycle C_k with edge e" is $(2n-k+1)$-stable for $5 \leq k \leq n$.*

2.5.6 "Pancyclicity"

From Theorem 2.21 we conclude that the property P: "G is pancyclic" is $(2n - 3)$-stable. Recently, Saito and Schiermeyer [67] have shown that $\frac{5}{6}n - 1 \leq s(P) \leq \frac{3}{2}n - 2$.

Theorem 2.32 (Faudree et al.) [31]. *The property P: "G is pancyclic" satisfies $\mathrm{cs}(P) = n + 1$ if n is even and $n \leq \mathrm{cs}(P) \leq n + 1$ if n is odd.*

A sketch of a proof for this result is as follows.

Suppose that G is not complete. By Theorem 2.25 $G-v$ has a Hamilton cycle C for every vertex $v \in V(G)$. Since $cl_{n+1}(G)$ is complete, G has maximum degree $\Delta(G) \geq (n + 1)/2$. Let $u \in V(G)$ be a vertex with $d(u) = (n + 1)/2$. Suppose there is no cycle of length k for some $3 \leq k \leq n - 1$, then u has no two neighbors on C at distance $k - 1$. A counting argument now implies that $d(u) \leq (n - 1)/2$, a contradiction.

Conjecture 2.33 (Schelten and Schiermeyer) [68]. *The property P: "G is pancyclic" satisfies $\mathrm{cs}(P) = n$ if n is odd.*

2.5.7 "Vertex-pancyclicity"

From Theorem 2.29 we conclude that "vertex-pancyclicity" is $(2n - 3)$-stable. However, it is still open whether this is best possible.

Theorem 2.34 (Randerath et al.) [63]. *The property P "G is vertex-pancyclic" satisfies $\mathrm{cs}(P) = \lceil \frac{4n-3}{3} \rceil$.*

2.5.8 "Edge-pancyclicity"

From Theorem 2.31 we conclude that "edge-pancyclicity" is $(2n-3)$-stable. However, it is still open whether this is best possible.

Theorem 2.35 (Randerath et al.) [63]. *The property P "G is edge-pancyclic" satisfies* $\mathrm{cs}(P) = \lceil \frac{3n-3}{2} \rceil$.

Corollary 2.36 [63]. *If $\mathrm{cl}_k(G)$ is complete for some $k \geq \lceil \frac{3n-3}{2} \rceil$, then every pair of vertices $u, v \in V(G)$ is connected by a path of length p for $2 \leq p \leq n-1$, i.e. G is* panconnected.

2.5.9 "G is cycle extendable"

A cycle C in a graph G is *extendable (in G)* if there exists a cycle C' in G such that $V(C) \subset V(C')$ and $|V(C')| = |V(C)|+1$. A graph is *cycle extendable* if G contains at least one cycle and every nonhamiltonian cycle in G is extendable.

Theorem 2.37 (Hendry) [45]. *The property P: "G is cycle extendable" satisfies $s(P) = 2n-4$.*

Theorem 2.38 (Randerath et al.) [63]. *The property P: "G is cycle extendable" satisfies* $\mathrm{cs}(P) = \frac{3}{2}n - 2$.

2.5.10 "G is fully cycle extendable"

A graph G is *fully cycle extendable* if G is cycle extendable and every vertex of G lies on a triangle of G. From Theorem 2.34 we obtain the following corollary.

Corollary 2.39 [63]. *The property P: "G is fully cycle extendable" satisfies* $\mathrm{cs}(P) = \frac{3}{2}n - 2$.

2.5.11 "G contains a path P_k"

Theorem 2.40 (Bondy and Chvátal) [9]. *The property P: "G contains a path P_k" satisfies $s(P) = n-1$ for $4 \leq k \leq n$.*

This was improved by Zhu, Tian and Deng [79].

Theorem 2.41 [79]. *Let $p = n-1-d(u)-d(v)$ and $T' = \{w \in T | d(w) \geq t + (n-k-1) + \max\{2, p\}\}$. If $d(u)+d(v) \geq n-1-t'$, then G contains a path P_k if $G+uv$ contains a path P_k.*

Theorem 2.42 (Faudree et al.) [31]. *The property P: "G contains a P_4" satisfies* $\sqrt{8n+9} - 3 \leq \mathrm{cs}(P) \leq \sqrt{8n+26} - 3$.

2.5.12 "G is k-Hamilton-connected"

A graph G is called *k-Hamilton-connected* if the deletion of at most k vertices from G results in a Hamilton-connected graph (i.e. every pair of vertices in the resulting graph is connected by a Hamilton path).

Theorem 2.43 (Bondy and Chvátal) [9]. *The property P: "G is k-Hamilton-connected" satisfies $s(P) = n + k + 1$.*

This was improved by Zhu, Tian and Deng [79].

Theorem 2.44 [79]. *Let k, n be positive integers with $k \leq n - 4$ and $T' = \{w \in T | d(w) \geq n - 3\}$. If $d(u) + d(v) \geq n + k + 1 - t'$, then G is k-Hamilton-connected if $G + uv$ is k-Hamilton-connected.*

Theorem 2.45 (Faudree et al.) [31]. *The property P: "G is k-Hamilton-connected" satisfies* $\mathrm{cs}(P) = n + k + 1$.

Limit example: $K_{k+2} + (K_{n-k-3} \cup K_1)$.

2.5.13 "G is k-leaf-connected"

A graph G is *k-leaf-connected* if $k < n$ and given any subset S of $V(G)$ with $|S| = k$, G has a spanning tree F such that the set S is the set of endvertices (leafs) of F. Thus a graph is *2-leaf-connected* if and only if it is Hamilton-connected. This generalization is due to Murty.

Theorem 2.46 (Gurgel and Wakabayashi) [39]. *The property P: "G is k-leaf-connected" satisfies $s(P) = n + k - 1$.*

Limit example: $K_k + (K_{\lfloor \frac{n-k}{2} \rfloor} \cup K_{\lfloor \frac{n-k+1}{2} \rfloor})$.

2.5.14 "G contains a $K_{2,k}$ $(2 \leq k \leq n-2)$"

Theorem 2.47 (Bondy and Chvátal) [9]. *The property P: "G contains a $K_{2,k}$ $(2 \leq k \leq n-2)$" satisfies $s(P) = n + k - 2$.*

This was improved by Zhu, Tian and Deng [79].

Theorem 2.48 [79]. *If $d(u) + d(v) \geq n + k - 2 - t$, then G contains $K_{2,k}$ if $G + uv$ contains $K_{2,k}$.*

This was further improved by Schiermeyer [72].

Theorem 2.49 [72]. *Let $m = \max\{\lambda_{uv}, \{\lambda_{uw}, \lambda_{vw} | w \in T\}\}$. If $m \geq k$, then G contains $K_{2,k}$ if $G + uv$ contains $K_{2,k}$.*

Theorem 2.50 (Faudree et al.) [31]. *The property P: "G contains a $K_{2,k}$" satisfies* $\sqrt{8n+9} - 4 \leq \mathrm{cs}(P) \leq \sqrt{8(k-1)n}$.

2.5.15 "G contains a kK_2 $(2k \leq n)$"

Theorem 2.51 (Bondy and Chvátal) [9]. *The property P: "G contains a kK_2 $(2k \leq n)$" satisfies $s(P) = 2k - 1$.*

This was improved by Zhu, Tian and Deng [79].

Theorem 2.52 [79]. *Let $R' = \{w \in T | d(w) \geq 2k - 1\}$. If $d(u) + d(v) \geq 2k - 1 - |R'|$, then G contains kK_2 if $G + uv$ contains kK_2.*

This was further improved by Schiermeyer [72].

Theorem 2.53 [72]. *If $k - \lambda_{uv} \leq \frac{n-1}{2}$ and $d(u) + d(v) + d(w) \geq 2k - 1 + \lambda_{uvw}$ for at least $2k + t + 1 - n - \lambda_{uv}$ vertices $w \in T$, where $\lambda_{uvw} := |N(u) \cap N(v) \cap N(w)|$, then G contains kK_2 if $G + uv$ contains kK_2.*

Theorem 2.54 (Faudree et al.) [31]. *The property P: "G contains a kK_2" satisfies* $\mathrm{cs}(P) = 2k - 1$.

Limit example: $K_{k-1} + \overline{K_{n-k+1}}$.

2.5.16 "G is k-factor-critical"

For a nonnegative integer k, a graph G is said to be *k-factor-critical* if, for any set $S \subset V(G)$ with $|S| = k$, the graph $G - S$ has a perfect matching (or, equivalently, every induced subgraph of order $n - k$ of G has a perfect matching).

The next two theorems are due to Plummer and Saito [61].

Theorem 2.55 [61]. *The property P: "G is k-factor-critical" satisfies $s(P) \leq n + k - 1$.*

Theorem 2.56 [61]. *Let G be a p-connected graph. If $|N(u) \cup N(v)| \geq n - p + k - 1$, then G is k-factor critical if $G + uv$ is k-factor-critical.*

2.5.17 "G is k-matching-extendable"

For an integer k, $0 \leq k \leq |V(G)|/2$, a graph G of even order is *k-matching-extendable* if G has a matching of size k and every matching of size k can be extended to (i.e. is a subgraph of) a perfect matching of G. Note that 0-matching-extendable is equivalent to having a perfect matching and, clearly, every $2p$-factor-critical graph is p-matching-extendable.

The next two theorems are due to Plummer and Saito [61].

Theorem 2.57 [61]. *The property P: "G is k-matching-extendable" satisfies $s(P) \leq n + 2k - 1$.*

Theorem 2.58 [61]. *Let G be a p-connected graph. If $|N(u) \cup N(v)| \geq n - p + 2k - 1$, then G is k-matching-extendable if $G + uv$ is k-matching-extendable.*

2.5.18 "G contains a k-factor $(2 \le k \le n)$"

Theorem 2.59 (Bondy and Chvátal) [9]. *The property P: "G contains a k-factor (kn even)" satisfies $s(P) = n + 2k - 4$.*

This was improved by Zhu, Tian and Deng [79].

Theorem 2.60 [79]. *Let n, k be positive integers with $2 \le k < n$. Let $T' = \{w \in T | d(w) \ge t + 3k - 4\}$. If $d(u) + d(v) \ge n + 2k - 4 - t'$, then G has a k-factor if $G + uv$ has a k-factor.*

Theorem 2.61 (Faudree et al.) [31]. *The property P: "G contains a k-factor ($2 \le k \le n - 1$, kn even)" satisfies $n + k - 2 \le \mathrm{cs}(P) \le n + 2k - 4$.*

This was improved by Niessen [58].

Theorem 2.62 [58]. *The property P: "G contains a k-factor ($1 \le k \le n-1$, kn even)" satisfies $\mathrm{cs}(P) = n + k - 2$ for $1 \le k \le 3$ and $n + k - 2 \le \mathrm{cs}(P) \le n + k - 1$ for $k \ge 4$.*

Niessen [58] conjectures that $\mathrm{cs}(P) = n + k - 2$ also holds for $k \ge 4$.

2.5.19 "G is k-connected or k-edge-connected"

Theorem 2.63 (Bondy and Chvátal) [9]. *The property P: "G is k-connected or k-edge-connected" satisfies $s(P) = n + k - 2$.*

This was improved by Zhu, Tian and Deng [79].

Theorem 2.64 [79]. *If $k \le n - 2$ and $d(u) + d(v) \ge n + k - 2 - t$, then G is k-connected or k-edge-connected if $G + uv$ is k-connected or k-edge-connected, respectively.*

This was further improved by Schiermeyer [72].

Theorem 2.65 [72]. *Let $k \le n - 2$. If $\lambda_{uv} + t \ge k$ or if $\lambda_{uv} < k$ and $\lambda_{uv} - \lambda_{uvw} + \min\{\lambda_{uw}, \lambda_{vw}\} \ge k$ for at least $k - \lambda_{uv}$ vertices $w \in T$, then G is k-connected or k-edge-connected if $G + uv$ is k-connected or k-edge-connected, respectively.*

Theorem 2.66 (Faudree et al.) [31]. *The property P: "G is k-connected or k-edge-connected" satisfies $\mathrm{cs}(P) = n + k - 2$.*

Limit example: $K_{k-1} + (K_{n-k} \cup K_1)$.

2.5.20 "$\alpha(G) \le k$"

Theorem 2.67 (Bondy and Chvátal) [9]. *The property P: "$\alpha(G) \le k$" satisfies $s(P) = 2n - 2k - 1$.*

This was improved by Zhu, Tian and Deng [79].

Theorem 2.68 [79]. *Let $R' = \{w \in T | d(w) \geq n-k\}$. If $d(u)+d(v) \geq 2n-2k-1-|R'|$, then $\alpha(G) \leq k$ if $\alpha(G+uv) \leq k$.*

This was further improved by Schiermeyer [72].

Theorem 2.69 [72]. *If $|N(u) \cup N(v) \cup N(w)| \geq n-k$ for at least $n-k-|N(u) \cup N(v)|$ vertices $w \in T$, then $\alpha(G) \leq k$ if $\alpha(G+uv) \leq k$.*

Theorem 2.70 (Faudree et al.) [31]. *The property P: "$\alpha(G) \leq k$" satisfies* $\mathrm{cs}(P) = 2n-2k-1$.

Limit example: $K_{n-k-1} + \overline{K_{k+1}}$.

2.5.21 "$\mu(G) \leq k$"

The smallest number of pairwise disjoint paths covering all the vertices of a graph G is denoted by $\mu(G)$.

Theorem 2.71 (Bondy and Chvátal) [9]. *The property P: "$\mu(G) \leq k$" satisfies $s(P) = n-k$.*

This was improved by Zhu, Tian and Deng [79].

Theorem 2.72 [79]. *Let $p = n-k-d(u)-d(v)$ and $T' = \{w \in T | d(w) \geq t-k+\max\{2,p\}\}$. If $d(u)+d(v) \geq n-k-t'$, then $\mu(G) \leq k$ if $\mu(G+uv) \leq k$.*

This was further improved by Schiermeyer [72].

Theorem 2.73 [72]. *If $d(u)+d(v)+d(w) \geq n-k+\lambda_{uv}$ for at least $\min\{t, t+2-\lambda_{uv}-k\}$ vertices $w \in T$, then $\mu(G) \leq k$ if $\mu(G+uv) \leq k$.*

Theorem 2.74 (Faudree et al.) [31]. *The property P: "$\mu(G) \leq k$" satisfies* $\mathrm{cs}(P) = n-k$.

Limit example: $K_{n-k} \cup \overline{K_k}$.

2.5.22 "G contains a clique K_t"

The stability of this property was not known so far. However, the graph $2K_1+(K_{t-2} \cup K_{n-t})$ shows that $s(P) = 2n-3$.

Theorem 2.75. *The property P: "G contains a clique K_t" satisfies $s(P) = 2n-3$.*

Theorem 2.76 (Faudree et al.) [31]. *The property P: "G contains a clique K_t" satisfies* $\mathrm{cs}(P) = 2\lfloor((t-2)/(t-1))n\rfloor + 1$.

2.5.23 "G contains every tree on k vertices"

Theorem 2.77 (Faudree et al.) [31]. *The property P: "G contains every tree on k vertices" satisfies $\sqrt{(k-2)n}/\sqrt{2} \leq \mathrm{cs}(P) \leq 2\sqrt{2(k-2)n}$.*

2.6 Miscellaneous

Amar et al. [3] have studied closure properties for balanced bipartite graphs, i.e. bipartite graphs whose partite sets have equal size. They introduced the *k-biclosure* of a balanced bipartite graph with color classes A and B as the graph obtained from G by recursively joining pairs of nonadjacent vertices respectively taken in A and B whose degree sum is at least k, until no such pair remains. A property P defined on all balanced bipartite graphs of order $2n$ is *k-bistable* if whenever $G + ab$ has property P and $d_G(a) + d_G(b) \geq k$, then G itself has property P. For several properties P in balanced bipartite graphs they have determined the bistability of P.

Möhring [56] has introduced a closure concept for asteroidal triple-free graphs.

Brandt [11], [13] has invented a closure for triangle-free graphs turning them into maximal triangle-free graphs. If the minimum degree of a graph is at least $n/3$, then its closure is unique and the independence number as well as the length of a longest cycle are stable.

Very recently, Stacho [75] has derived a new closure theorems for some graph properties related to cycles, which generalize the corresponding results of Bondy and Chvátal [9].

In several proofs for a sufficient condition for hamiltonian graphs a closure operation has been applied, e.g., in [4] and in [27].

In [41], Hanson defines a closure operation for the class of domination critical graphs.

3 Structural conditions

In the previous section we have seen a large number of closure results based on degree conditions and neighborhood conditions involving bounds related to the number of vertices of the graph or of certain subsets of the vertex set of the graph. These results all involve what we call global parameters of the graph, e.g., the number of vertices of the graph or the cardinality of a maximum independent set containing a special pair of vertices.

In this section we focus on closure results based on more local conditions concerning the extended neighborhood structure and certain subgraphs. One advantage of such conditions compared to the global ones is that the associated closure results can also be applied to graphs that are not necessarily very dense. A disadvantage is that we often lose the nice property of uniqueness of the closure. The special cases of these structural conditions for claw-free graphs are postponed to the next section.

3.1 Local closures

In an attempt to find closure results which are also applicable to large classes of graphs with few edges (linear to the number of vertices) and large diameter, Hasratian and Khachatrian [42],[43] introduced a number of closure concepts based on conditions concerning the extended neighborhood structure. Recall that, for a positive integer k and a vertex u of G, $N^k(u)$ and $M^k(u)$ denote the sets of all vertices $v \in V(G)$ with distance $d(u,v) = k$ and $d(u,v) \leq k$ from u in G, respectively. The subgraph of G induced by $M^k(u)$ is denoted by $G_k(u)$.

The main part of [42] deals with sufficient degree conditions for hamiltonicity involving the above extended neighborhood sets. At the end of [42] the authors observe the following: if $G + uv$ is hamiltonian, where u and v are two vertices of G with $d(u,v) = 2$, then G has a Hamilton path $v_1v_2\ldots v_n$ with $u = v_1$ and $v = v_n$. Let $N(u) = \{v_{i_1}, \ldots, v_{i_t}\}$. If G is not hamiltonian, then, using the standard transformation, it is clear that $vv_{i_j+1} \notin E$ for every j with $1 \leq j \leq t$, hence $d_{G_2(u)}(v) < |M^2(u)| - d(u)$. This proves the following local closure result.

Theorem 3.1 [42]. *Let u and v be two vertices of G such that $d(u,v) = 2$ and $d(u) + d_{G_2(u)}(v) \geq |M^2(u)|$. Then G is hamiltonian if and only if $G + uv$ is hamiltonian.*

In [43] the authors use similar observations to prove the following.

Theorem 3.2 [43]. *Let u and v be two vertices of G such that $d(u,v) = 2$ and $d(u) + d(v) \geq |M^2(u)| + |N^3(u) \cap N(v)|$. Then G is hamiltonian if and only if $G + uv$ is hamiltonian.*

Both results have the following consequence.

Corollary 3.3. *Let u and v be two vertices of G such that $d(u,v) = 2$ and $d(u)+d(v) \geq |M^3(u)|$. Then G is hamiltonian if and only if $G + uv$ is hamiltonian.*

Based on these results the authors defined several closures. Each of these closures need not be unique, but they all contain the Bondy-Chvátal n-closure $\mathrm{cl}_n(G)$ as a spanning subgraph, and G is hamiltonian if and only if the closure of G is hamiltonian. It is shown that for every $n \geq 6$ there exists a graph G with $|E| = 2n-3$ such that the closure based on Theorem 3.2 is a complete graph. Since the transformation used in the proof of Theorem 3.2 is exactly the same as in the Bondy-Chvátal closure result, all remarks on algorithmic aspects and complexity pertain to this local closure concept.

In [43] the authors give similar results for other graph properties. We omit the details. In [42] they give sufficient conditions for hamiltonicity based on extended neighborhood structures. We note here that, in contrast to the situation for the classic degree conditions, the sufficient conditions do not imply that the associated closures are complete.

3.2 Subgraph closures

Broersma [17] gave the first closure result based on a condition involving the neighborhood structure of a subgraph on four vertices. He called a pair of vertices of a graph G a K_4-*pair* if they are the two nonadjacent vertices of an induced subgraph of G isomorphic to K_4 minus an edge, and the other pair of vertices its *copair*. If G contains a Hamilton path $v_1 v_2 \ldots v_n$ between two vertices $u = v_1$ and $v = v_n$ of a K_4-pair, and if its copair $\{x, y\}$ satisfies $N(x) \cup N(y) \subseteq N[u] \cup N[v]$, then it is not difficult to show that G contains a Hamilton cycle: Assume without loss of generality that $x = v_i$ and $y = v_j$ for some i and j with $2 \leq i < j \leq n-1$. If $i = j-1$, then a Hamilton cycle is immediate from the same transformation as in the Bondy-Chvátal n-closure. By similar arguments, $v_{i+1} \notin N(u)$ and $v_{j-1} \notin N(v)$. Hence $i < j-2$, $v_{i+1} \in N(v)$, and $v_{j-1} \in N(u)$. But now $v_1 v_{j-1} v_{j-2} \ldots v_{i+1} v_n v_{n-1} \ldots v_j v_i v_{i-1} \ldots v_1$ is a Hamilton cycle in G. This proves the following main result in [17].

Theorem 3.4 [17]. *Let $\{u, v\}$ be a K_4-pair of G with copair $\{x, y\}$ such that $N(x) \cup N(y) \subseteq N[u] \cup N[v]$. Then G is hamiltonian if and only if $G + uv$ is hamiltonian.*

Based on Theorem 3.4, in [17] a graph H is called a K_4-*closure* of G if H can be obtained from G by recursively joining K_4-pairs satisfying the condition of Theorem 3.4, and if H contains no such pairs. As noted in [17] a graph can have different K_4-closures, but obtaining a K_4-closure of G can be helpful to answer the question whether G is hamiltonian, for instance, if K_n is a K_4-closure of G. We note here that the Bondy-Chvátal n-closure $\mathrm{cl}_n(G)$ of G need not be contained in some K_4-closure of G.

Theorem 3.5 [17]. *Let G be a graph on $n \geq 3$ vertices. If K_n is a K_4-closure of G, then G is hamiltonian.*

In [17] examples show that Theorem 3.5 can be useful in checking hamiltonicity or obtaining Hamilton cycles (in polynomial time) in cases where counterparts of Theorem 3.5 based on degree conditions or neighborhood conditions are not useful. Other examples in [17] show that Theorem 3.5 can be used to verify hamiltonicity of graphs with small maximum degree and large diameter, and also of graphs with many vertices of degree two.

Motivated by these results, and adopting ideas from [70], Broersma and Schiermeyer [19] obtained several results on local structure conditions similar to the above condition on K_4-pairs. We list their results here without proofs which are all similar to the given proof of Theorem 3.4, and we omit the counterparts of Theorem 3.5.

Theorem 3.6 [19]. *Let $\{x, y, u, v\}$ be a subset of four vertices of V such that $uv \notin E$ and $x, y \in N(u)$. If $N(x) \cup N(y) \subseteq N[v] \cup \{u\}$, then G is hamiltonian if and only if $G + uv$ is hamiltonian.*

Theorem 3.7 [19]. *Let $\{x, y, u, v\}$ be a subset of four vertices of V such that $uv \notin E$, $x \in N(u) \cap N(v)$, and $y \in N(u)$. If $N(y) \subseteq N(v) \cup \{u\}$ and $N(x) \subseteq N[y] \cup \{v\}$, then G is hamiltonian if and only if $G + uv$ is hamiltonian.*

Theorem 3.8 [19]. *Let $\{x, y, u, v\}$ be a subset of four vertices of V such that $uv \notin E$ and $x \in N(u)$. If $N(y) \subseteq N[v] \cup \{u\}$ and $N(x) \subseteq N[y] \cup \{v\}$, then G is hamiltonian if and only if $G + uv$ is hamiltonian.*

Theorem 3.9 [19]. *Let $\{x, y, u, v\}$ be a subset of four vertices of V such that $uv \notin E$ and $xy \in E$. If $N(x) \subseteq N[u] \cup \{v\}$ and $N(y) \subseteq N[v] \cup \{u\}$, then G is hamiltonian if and only if $G + uv$ is hamiltonian.*

Examples given in [19] show that the conditions in Theorems 3.4, 3.6, 3.7, 3.8, 3.9 are sharp in the following sense. For each of the conditions one can find a nonhamiltonian graph G containing four vertices $\{u, v, x, y\}$ with $uv \notin E$ such that $G + uv$ is hamiltonian and such that precisely one vertex of G violates the conditions whereas the other conditions are satisfied.

Clearly, for a given subset $\{u, v, x, y\}$ of $V(G)$ each of the conditions in Theorems 3.4, 3.6, 3.7, 3.8, 3.9 can be checked in polynomial time. Thus, as in [9], these results all lead to algorithms which construct a closure based on these conditions in polynomial time. Moreover, one easily checks from the transformations used in the proofs, that a Hamilton cycle in a closure based on these conditions can be transformed into a Hamilton cycle of G in polynomial time.

In the previous section we saw that all graphs satisfying one of the classic degree conditions have a complete Bondy-Chvátal n-closure. The situation is different for the conditions of Theorems 3.4, 3.6, 3.7, 3.8, 3.9. In [19] an example is given of a graph G in which all K_4-pairs satisfy the conditions of Theorem 3.4, but for which the K_4-closure need not be complete. Similar remarks can be made for the conditions of Theorems 3.6, 3.7, 3.8, 3.9. However, in [19] sufficiency counterparts of most of these results are proved directly. We omit the details.

One motivation for defining and studying the notion of K_4-pairs and K_4-closures in [17] was that in claw-free graphs, i.e. graphs that do not contain $K_{1,3}$ as an induced subgraph, K_4-pairs trivially satisfy the condition of Theorem 3.4. Of course adding edges to a claw-free graph might give rise to an induced $K_{1,3}$. In fact, it was shown in [17] that adding edges one by one to a claw-free graph satisfying the conditions of Theorem 3.4 can indeed yield an induced $K_{1,3}$. However, this can be avoided if one adopts the method established by Ryjáček [64], which will be described in the next section. Motivated by the above phenomena Broersma and Trommel [20] introduced a variation on the K_4-closure based on the following result.

Theorem 3.10 [20]. *Let $G = (V, E)$ be a graph and let $\{x, y, u, v\}$ be a subset of four vertices of V such that $uv \notin E$ and $\{x, y\} \subseteq N(u) \cap N(v)$. If $N(x) \subseteq N[u] \cup N[v]$ and $N(y) \setminus N[x]$ induces a complete graph (or is empty), then for every cycle C' of $G + uv$ there exists a cycle C of G such that $V(C') \subseteq V(C)$.*

Remark that x and y can be nonadjacent in Theorem 3.10, and the graph G need not be claw-free. Note, however, that if G is a claw-free graph, then the conditions of Theorem 3.10 are always satisfied if x and y are adjacent.

Based on Theorem 3.10, a graph H is called a K_4^**-closure* of a graph G if H can be obtained from G by iteratively joining pairs $\{u, v\}$ satisfying the conditions in Theorem 3.10 for some $\{x, y\} \subseteq N(u) \cap N(v)$, and if H contains no such pairs. Similar to the situation for the K_4-closure, a graph can have different K_4^*-closures. If G has a unique K_4^*-closure, then we denote it by $K_4^*(G)$. Examples in [20] show that there exists a graph G_1 such that $K_4(G_1) = G_1$ and $K_4^*(G_1) = G_1 + uv$ for two nonadjacent vertices u and v of G_1, as well as a graph G_2 such that $K_4^*(G_2) = G_2$ and $K_4(G_2) = G_2 + uv$ for two nonadjacent vertices u and v of G_2.

As an obvious consequence of Theorem 3.10, we obtain the following result.

Corollary 3.11. *For any graph G and any K_4^*-closure H of G, $c(H) = c(G)$.*

It was shown in [20] that for every claw-free graph G, the claw-free closure of G (See Section 4) is contained in some K_4^*-closure of G.

A similar, but different closure concept was obtained in [20] by combining the condition on K_4-pairs of Theorem 3.4 with the following condition on subgraphs isomorphic to K_5 minus an edge.

Theorem 3.12 [20]. *Let $G = (V, E)$ be a graph and let $\{x, y_1, y_2, u, v\}$ be a subset of five vertices of V such that $G[\{x, y_1, y_2, u, v\}] = K_5 - uv$. If*

(i) $N(x) \subseteq N[u] \cup N[v]$,

(ii) $N(y_i) \subseteq N[u] \cup N[v] \cup N(y_{3-i})$ $(i = 1, 2)$, and

(iii) $N(y_i) \setminus N[x]$ induces a complete graph (or is an empty set) $(i = 1, 2)$,

then for every cycle C' of $G + uv$ there exists a cycle C of G such that $V(C') \subseteq V(C)$.

Based on Theorem 3.12, a graph H is called a K_5*-closure* of a graph G if H can be obtained from G by iteratively joining pairs $\{u, v\}$ satisfying the condition in Theorem 3.12 for some $\{x, y_1, y_2\}$ such that $G[\{x, y_1, y_2, u, v\}] = K_5 - uv$, and if H contains no such pairs. Clearly, we get the following analogue of Corollary 3.11.

Corollary 3.13. *For any graph G and any K_5-closure H of G, $c(H) = c(G)$.*

Note that in a claw-free graph G the conditions (1), (2) and (3) of Theorem 3.12 are satisfied for any $\{x, y_1, y_2, u, v\} \subseteq V(G)$ with $G[\{x, y_1, y_2, u, v\}] = K_5 - uv$. In [20] it is shown how the claw-free closure (See Section 4) of a claw-free graph G can be obtained from G using a combination of the K_4-closure and K_5-closure.

4 Claw-free graphs

In this section we present a closure concept for the class of claw-free graphs, that was recently introduced by Ryjáček [64]. Some of the features of this concept are slightly different from those presented in the previous chapters. Namely, the claw-free closure in its original form typically adds more edges in one step of the construction. The resulting graph is uniquely determined and it can be shown that the closure operation turns a claw-free graph into a line graph of a triangle-free graph, preserving a number of cycle and path properties. This fact makes this concept a powerful tool for improving many known results on claw-free graphs. These applications are listed in Subsection 4.2. Since the concept is based on a condition of a local character, it is applicable also to graphs that are not very dense.

4.1 Definition and basic properties

A graph G is *claw-free* if G does not contain an induced subgraph isomorphic to the complete bipartite graph $K_{1,3}$ (referred to as *a claw*). For additional information on results on claw-free graphs we refer the reader to the survey paper [33].

For a vertex x of a graph G the subgraph of G induced by the set of neighbors $N_G(x)$ of x is called the *neighborhood* of x (in G). We say that x is *locally connected* if its neighborhood is a connected graph. A locally connected vertex with a noncomplete neighborhood is called *eligible* and the graph G'_x, obtained from G by adding to the neighborhood of an eligible vertex x all missing edges, (i.e. such that $N_{G'_x}(x)$ induces in G'_x a complete graph), is called the *local completion of G at x* . The basic observation for the next closure concept is given in the following statement (in which we denote by $c(G)$ the circumference of G).

Proposition 4.1 [64]. *Let G be a claw-free graph and let x be an eligible vertex of G. Let $N'_x = \{uv \mid u, v \in N_G(x), uv \notin E(G)\}$ and let G'_x be the graph with vertex set $V(G'_x) = V(G)$ and with edge set $E(G'_x) = E(G) \cup N'_x$. Then*
(*i*) *the graph G'_x is claw-free,*
(*ii*) $c(G'_x) = c(G)$.

It is not difficult to show that a graph which is obtained from a claw-free graph G by recursively performing the local completion operation as long as there is at least one eligible vertex, is uniquely determined and is again claw-free. This graph is called the *(claw-free) closure of G* and is denoted by $\mathrm{cl}^c(G)$.

Since $\mathrm{cl}^c(G)$ is claw-free and has no eligible vertices, $\mathrm{cl}^c(G)$ is the line graph of a triangle-free graph. We thus have the following theorem.

Theorem 4.2 [64]. *Let G be a claw-free graph. Then*
(*i*) $\mathrm{cl}^c(G)$ *is well-defined (i.e., uniquely determined by G),*
(*ii*) *there is a triangle-free graph H such that $\mathrm{cl}^c(G)$ is the line graph of H,*
(*iii*) $c(\mathrm{cl}^c(G)) = c(G)$,
(*iv*) *G is hamiltonian if and only if $\mathrm{cl}^c(G)$ is hamiltonian.*

If $G = \mathrm{cl}^c(G)$, then we say that the graph G is *closed* (thus, G is closed if and only if G is the line graph of a triangle-free graph). As mentioned in [64], $\mathrm{cl}^c(G)$ can be equivalently characterized as the smallest claw-free and $(K_4 - e)$-free graph containing the graph G as a spanning subgraph. It is thus natural to ask whether the closure $\mathrm{cl}^c(G)$ can be obtained by performing some of the closure operations presented in Section 3 under some additional conditions concerning the strategy of adding edges (due to the fact that these closures are not unique).

As shown in [17], the graph G, obtained from a cycle C_6 with vertices $x_1, \ldots, x_6$ by adding three edges x_2x_4, x_2x_6, x_4x_6, is claw-free and $\mathrm{cl^c}(G)$ is complete, while its K_4-closure is isomorphic to $K_6 - E(K_3)$. On the other hand, for the concepts of a K_4^*-closure and a K_5-closure (see Section 3), it was shown in [20] that, for any claw-free graph G, $\mathrm{cl^c}(G)$ can be obtained both as a spanning subgraph of some K_4^*-closure as well as by an appropriate combination of the K_4- and K_5-closure operations. Moreover, as pointed out in Section 3, the K_4^*-, K_4- and K_5-closures are applicable also for graphs containing claws. The drawback of this larger generality is the loss of the property of uniqueness (which, together with the fact that $\mathrm{cl^c}(G)$ is a line graph of a triangle-free graph, makes $\mathrm{cl^c}$ a powerful tool in applications).

As an example of an immediate application of the closure operation we can mention here the following result. Zhan [77] and independently Jackson [48] proved that every 7-connected line graph is hamiltonian. Thus, if there is a 7-connected nonhamiltonian claw-free graph G, then $\mathrm{cl^c}(G)$ is a 7-connected (since the connectivity cannot decrease by adding edges) nonhamiltonian line graph, which is impossible. We thus have the following result.

Theorem 4.3 [64]. *Every 7-connected claw-free graph is hamiltonian.*

More general results were obtained by Li [55] and Fan [30] (in [55], Theorem 4.3 is extended to 6-connected graphs with at most 33 vertices of degree 6). Further applications will be shown in Subsections 4.2.1 and 4.2.2.

In the terminology of Section 2.1, parts (*iii*), (*iv*) of Theorem 4.2 can be equivalently stated to read that the value of $c(G)$ and the property of being hamiltonian are stable under $\mathrm{cl^c}$.

The following theorem summarizes known results concerning the stability of other graph properties with respect to the closure operation. We denote here by $p(G)$ the length of a longest path in G.

Theorem 4.4 [14], [46], [65]. *Let G be a claw-free graph. Then*

(*i*) [14] $p(\mathrm{cl^c}(G)) = p(G)$,

(*ii*) [14] *G is traceable if and only if $\mathrm{cl^c}(G)$ is traceable,*

(*iii*) [65] *G can be covered by k cycles if and only if $\mathrm{cl^c}(G)$ can be covered by k cycles,*

(*iv*) [65] *G has a 2-factor with at most k components if and only if $\mathrm{cl^c}(G)$ has a 2-factor with at most k components,*

(*v*) [46] *G can be covered by k paths if and only if $\mathrm{cl^c}(G)$ can be covered by k paths,*

(*vi*) [46] *G has a path-factor with k components if and only if $\mathrm{cl^c}(G)$ has a path-factor with k components.*

On the other hand, it was shown in [14] that the properties of (vertex) pancyclicity and (full) cycle extendability are not stable (even under the additional assumption of arbitrarily high connectivity).

Theorem 4.5 [14]. *Let $k \geq 2$ be an arbitrary integer. Then*
- (i) *there is a k-connected claw-free graph G such that G is not pancyclic, but $\mathrm{cl}^{\mathrm{c}}(G)$ is vertex pancyclic;*
- (ii) *there is a k-connected claw-free graph G such that G is not cycle extendable, but $\mathrm{cl}^{\mathrm{c}}(G)$ is fully cycle extendable.*

Niessen [59] gave examples showing that the property to have a k-factor is not stable for $k = 4, 6$ and $k \geq 8$.

By Theorem 4.5(i), there are many nonpancyclic claw-free graphs with a pancyclic closure. This fact motivates a natural question: how many cycle lengths can be missing in a claw-free graph with a pancyclic (or, specifically, complete) closure? In [66], it has been proved that every claw-free graph with a complete closure has a cycle of length $n-1$, and the following conjecture was stated.

Conjecture 4.6 [66]. *Let G be a claw-free graph of order n whose closure is complete and let c_1 and c_2 be fixed constants. Then for sufficiently large n, the graph G contains cycles C_i for all i, with $3 \leq i \leq c_1$ and $n - c_2 \leq i \leq n$.*

A graph G is said to be *homogeneously traceable* if each of its vertices is an endvertex of some Hamilton path in G. Obviously, every homogeneously traceable graph is 2-connected.

Consider the graph G_1 in Figure 4.1 (where the circular parts represent complete graphs of appropriate order).

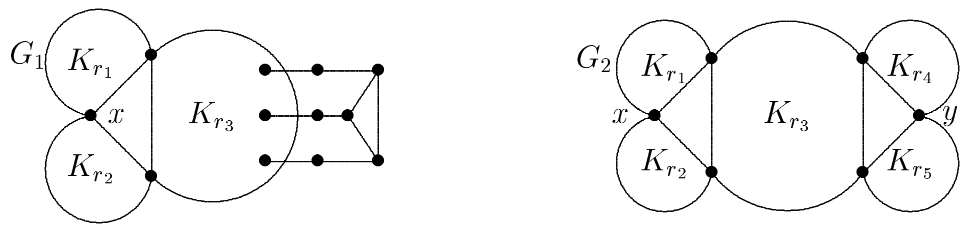


Figure 4.1

Then G_1 is 2-connected and has no Hamilton path with endvertex x, but it is easy to check that $\mathrm{cl}^{\mathrm{c}}(G_1)$ is homogeneously traceable. On the other hand, by Theorem 4.3, homogeneous traceability is trivially stable in the class of 7-connected claw-free graphs.

The graph G_2 in Figure 4.1 has no Hamilton path joining the vertices x, y, and $\mathrm{cl}^{\mathrm{c}}(G)$ is complete (and hence Hamilton-connected). It should be noted that G_2 has connectivity 2, while Hamilton-connectedness implies connectivity at least 3; however, no such 3-connected example is known.

Brandt [12] introduced the following extension of the claw-free closure. For any set $S \subset V(G)$ of vertices of a claw-free graph denote by $\mathrm{cl}^{\mathrm{c}}(G, S)$ the supergraph of G obtained by successively adding missing edges (one edge at a step) to those induced $(K_4 - e)$'s, both vertices of degree 3 of which are outside S. It was shown in [12] that (i) $\mathrm{cl}^{\mathrm{c}}(G, S)$ is uniquely determined by G and by $S \subset V(G)$, (ii) for any $a, b \in V(G)$, the length of a longest (a, b)-path remains unchanged in $\mathrm{cl}^{\mathrm{c}}(G, \{a, b\})$, (iii) all induced claws in $\mathrm{cl}^{\mathrm{c}}(G, \{a, b\})$ are centered in $\{a, b\}$. From these facts Brandt obtained the following result.

Theorem 4.7 [12]. *Every 9-connected claw-free graph is Hamilton-connected.*

This result motivates the following question.

Problem 4.8.

(i) *Determine the smallest positive integer k_1 such that, for every k_1-connected claw-free graph G, the graph G is homogeneously traceable if and only if $\mathrm{cl}^{\mathrm{c}}(G)$ is homogeneously traceable.*

(ii) *Determine the smallest positive integer k_2 such that, for every k_2-connected claw-free graph G, the graph G is Hamilton-connected if and only if $\mathrm{cl}^{\mathrm{c}}(G)$ is Hamilton-connected.*

From Theorems 4.3 and 4.7 and by the previous examples, we know that $3 \leq k_1 \leq 7$ and $3 \leq k_2 \leq 9$.

Another approach to the question of stability of homogeneous traceability and Hamilton-connectedness was used in [7]. A vertex $x \in V(G)$ is *locally k-connected* $(k \geq 1)$, if $N(x)$ induces a k-connected graph. A locally k-connected vertex with a noncomplete neighborhood is said to be *k-eligible*. The *(claw-free) k-closure* of a claw-free graph G is the graph $\mathrm{cl}^{\mathrm{c}}_k(G)$, obtained from G by recursively performing the local completion operation at k-eligible vertices, as long as there is at least one k-eligible vertex. The following results were proved in [7].

Theorem 4.9 [7]. *Let G be a claw-free graph. Then*

(i) $\mathrm{cl}^{\mathrm{c}}_k(G)$ *is unique for any* $k \geq 1$,

(ii) *G is homogeneously traceable if and only if $\mathrm{cl}^{\mathrm{c}}_2(G)$ is homogeneously traceable,*

(iii) *G is Hamilton-connected if and only if $\mathrm{cl}^{\mathrm{c}}_3(G)$ is Hamilton-connected.*

A graph G is *$(2, k)$-factor-critical* if, for every set $X \subset V(G)$ with $|X| = k$, the graph $G - X$ has a 2-factor. Ishizuka [47] and Plummer and Saito [61] proved the following results.

Theorem 4.10 [47], [61]. *Let G be a claw-free graph and $k \geq 1$ an integer. Then*

(i) [61] *G is k-factor-critical if and only if $\mathrm{cl}^{\mathrm{c}}_k(G)$ is k-factor-critical,*

(ii) [47] *G is $(2, k)$-factor-critical if and only if $\mathrm{cl}^{\mathrm{c}}_{k+1}(G)$ is $(2, k)$-factor-critical.*

Concerning k-matching-extendability, it is shown in [61] that if $\mathrm{cl}^{\mathrm{c}}_{2k}(G)$ is k-matching-extendable, then so is G, but, surprisingly, the converse does not hold for $k \geq 0$.

4.2 Applications

4.2.1 Forbidden subgraph conditions for hamiltonicity

If $H_1, \ldots, H_k$ are graphs, then a graph G is said to be $(H_1, \ldots, H_k)$-*free*, if G does not contain an induced subgraph isomorphic to any of the graphs $H_1, \ldots, H_k$. The graphs $H_1, \ldots, H_k$ are referred to as *forbidden induced subgraphs*. Graphs that will be used as forbidden induced subgraphs are listed in Figure 4.2. We will further denote by C a claw, by T a triangle and by P_k a path on k vertices.

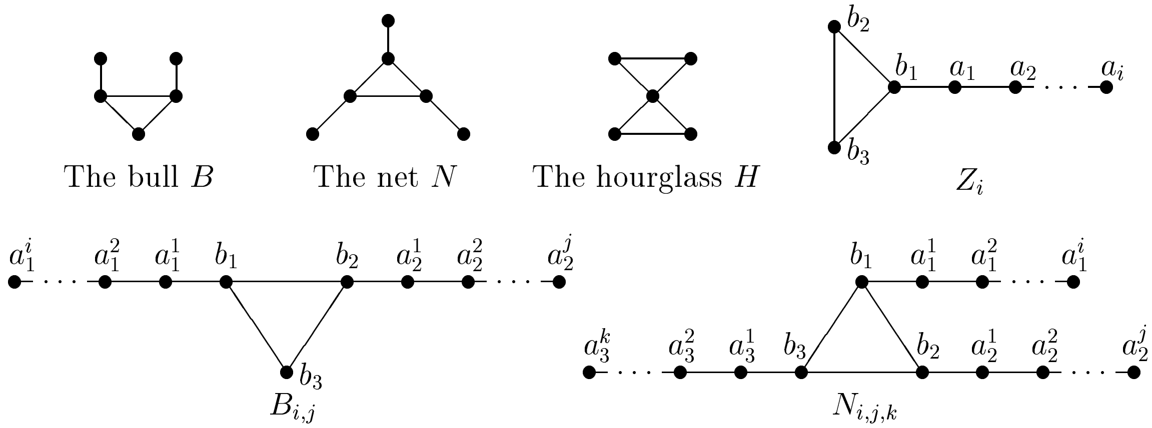
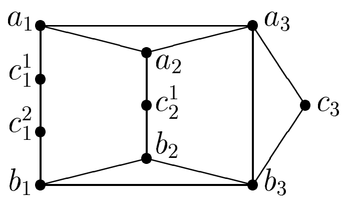


Figure 4.2

Bedrossian [5] characterized all pairs of connected forbidden subgraphs X, Y such that every 2-connected (X, Y)-free graph is hamiltonian.

Theorem 4.11 [5]. *Let X and Y be connected graphs with $X,\ Y \neq P_3$, and let G be a 2-connected graph that is not a cycle. Then, G being (X, Y)-free implies G is hamiltonian if and only if (up to symmetry) $X = C$ and $Y = P_4, P_5, P_6, C_3, Z_1, Z_2, B_{1,1}, B_{1,2}$ or $N_{1,1,1}$.*

Following [21], we denote by $\mathcal{P}$ the class of all graphs that are obtained by taking two vertex-disjoint triangles with vertex sets $\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$ and by joining every pair of vertices $\{a_i, b_i\}$ by a path $P_{k_i} = a_i c_i^1 c_i^2 \ldots c_i^{k_i-2} b_i$ for some $k_i \geq 3$ or by a triangle with vertex set $\{a_i, b_i, c_i\}$. We denote a graph from $\mathcal{P}$ by P_{x_1,x_2,x_3}, where $x_i = k_i$ if a_i, b_i are joined by a P_{k_i}, and $x_i = T$, if a_i, b_i are joined by a triangle (see Figure 4.3).



$P_{4,3,T} \in \mathcal{P}$

Figure 4.3

Observing that $P_{T,T,T}$ and $P_{3,T,T}$ are the only two 2-connected nonhamiltonian CZ_3-free graphs, Faudree and Gould [34] added the graph Z_3 to the list of Theorem 4.11, under the additional assumption that $n \geq 10$.

A class $\mathcal{C}$ of graphs such that $\mathrm{cl^c}(G) \in \mathcal{C}$ for every $G \in \mathcal{C}$ is called a *stable class.* Clearly, the class of (C, A)-free graphs is trivially stable if A is not claw-free or if A is not closed. The following theorem characterizes all connected closed claw-free graphs A for which the (C, A)-free class is stable.

Theorem 4.12 [23]. *Let A be a closed connected claw-free graph. Then G being (C, A)-free implies $\mathrm{cl^c}(G)$ is (C, A)-free if and only if*

$$A \in \{H, T\} \cup \{P_i |\ i \geq 3\} \cup \{Z_i |\ i \geq 1\} \cup \{N_{i,j,k} |\ i, j, k \geq 1\}.$$

Brousek [21] proved that every 2-connected nonhamiltonian claw-free graph contains an induced subgraph from the class $\mathcal{P}$. Using this result, Theorem 4.12 and the special structural properties of closed claw-free graphs, it is possible to prove the following results, extending the "if" part of Theorem 4.11. The classes of graphs that occur in Theorem 4.13 as classes of nonhamiltonian exceptions, are shown in Figure 4.4 (where the elliptical parts represent cliques of appropriate order). It should be noted here that, for the sake of brevity, some of the exceptional classes are merged to a larger one here and, in these cases, a bit more can be said about their structure. Interested readers can find more information in the respective papers [22], [23].

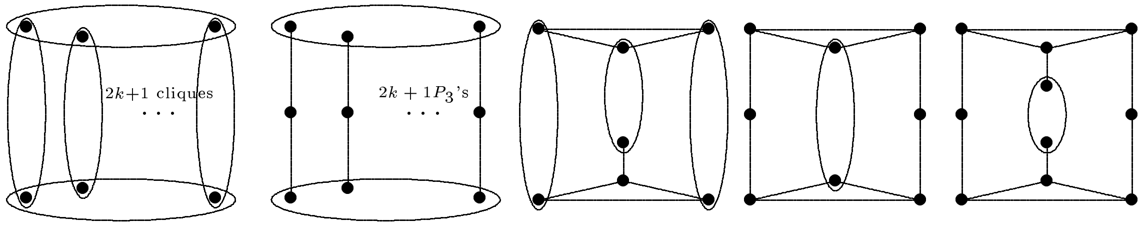


Figure 4.4

Theorem 4.13 [22], [23].

(*i*) [22] *If G is a 2-connected (C, P_7)-free graph, then either G is hamiltonian or $\mathrm{cl^c}(G) \in \mathcal{F}_1^k$ for some $k \geq 1$.*

(*ii*) [22] *If G is a 2-connected (C, Z_4)-free graph, then either G is hamiltonian, or $G \in \{P_{3,T,T}, P_{3,3,T}, P_{3,3,3}, P_{4,T,T}\}$, or $\mathrm{cl^c}(G) \in \mathcal{F}_1^k$ for some $k \geq 1$.*

(*iii*) [22] *If G is a 2-connected $(C, N_{1,2,2}, N_{1,1,3})$-free graph, then either G is hamiltonian, or $G \simeq P_{3,3,3}$, or $\mathrm{cl^c}(G) \in \mathcal{F}_1^1 \cup \mathcal{F}_3 \cup \mathcal{F}_4$.*

(*iv*) [22] *If G is a 2-connected $(C, N_{1,1,2})$-free graph, then either G is hamiltonian or $G \in \mathcal{F}_1^1$.*

(*v*) [23] *If G is a 2-connected (C, H, P_8)-free graph, then either G is hamiltonian or* $\mathrm{cl}^{\mathrm{c}}(G) \in \mathcal{F}_2^k$ *for some* $k \geq 1$.

(*vi*) [23] *If G is a 2-connected (C, H, Z_5)-free graph, then either G is hamiltonian, or* $G \simeq P_{4,3,3}$*, or* $\mathrm{cl}^{\mathrm{c}}(G) \in \mathcal{F}_2^k$ *for some* $k \geq 1$.

(*vii*) [23] *If G is a 2-connected $CHN_{1,1,4}$-free graph, then either G is hamiltonian or* $\mathrm{cl}^{\mathrm{c}}(G) \in \mathcal{F}_2^1 \cup \mathcal{F}_5$.

Note that all the nonhamiltonian exceptions in Theorem 4.13 have connectivity 2 and thus, in each of the classes of Theorem 4.13, 3-connectedness implies hamiltonicity.

4.2.2 Degree conditions for hamiltonicity

Using the structural properties of closed claw-free graphs, it is possible to prove that a nonhamiltonian closed claw-free graph with large degrees can be covered by relatively few cliques. This fact was observed by Favaron, Flandrin, Li and Ryjáček [37] and stated as follows (where we denote by $\theta(G)$ the clique covering number of G).

Theorem 4.14 [37]. *Let $k \geq 4$ be an integer and let G be a 2-connected claw-free graph with $|V(G)| = n$ such that $n \geq 3k^2 - 4k - 7$, $\delta(G) \geq 3k - 4$ and*

$$\sigma_k(G) > n + k^2 - 4k + 7.$$

Then either $\theta(\mathrm{cl}^{\mathrm{c}}(G)) \leq k - 1$ or G is hamiltonian.

Specifically, Theorem 4.14 implies that, for any integer $k \geq 4$, the closure of every nonhamiltonian claw-free graph G with $n \geq 3k^2 - k - 4$ and $\delta(G) > \frac{n+(k-2)^2}{k}$ can be covered by at most $k - 1$ cliques. This implies that for proving a minimum degree condition for hamiltonicity of type $\delta(G) > \frac{n}{k} + c$ for any given $k \geq 4$, it is enough to list all nonhamiltonian closed claw-free graphs with $\theta(G) \leq k - 1$.

A characterization of closed nonhamiltonian claw-free graphs with small clique covering number can be achieved by using the correspondence between the graphs and their line graph preimages. Namely, the following was proved for $\theta \leq 5$ in [37] and independently by Kuipers and Veldman in [52]. The classes $\mathcal{F}_6, \ldots, \mathcal{F}_{10}$ are shown in Figure 4.5 (where the elliptical parts represent again cliques of appropriate order).

Theorem 4.15 [37], [52]. *Let G be a 2-connected closed claw-free graph.*

(*i*) *If $\theta(G) \leq 2$, then G is hamiltonian.*

(*ii*) *If $3 \leq \theta(G) \leq 5$, then either G is hamiltonian or G is a spanning subgraph of a graph from* $(\cup_{i=1}^{2} \mathcal{F}_1^i) \cup (\cup_{i=6}^{10} \mathcal{F}_i)$.

Combining Theorem 4.14 (for $k = 6$) and Theorem 4.15, we now obtain the following result.

Corollary 4.16 [37]. *Let G be a 2-connected claw-free graph with $n \geq 77$ vertices such that $\delta(G) \geq 14$ and $\sigma_6(G) > n + 19$. Then either G is hamiltonian or G is a spanning subgraph of a graph from* $(\cup_{i=1}^{2} \mathcal{F}_1^i) \cup (\cup_{i=6}^{10} \mathcal{F}_i)$.

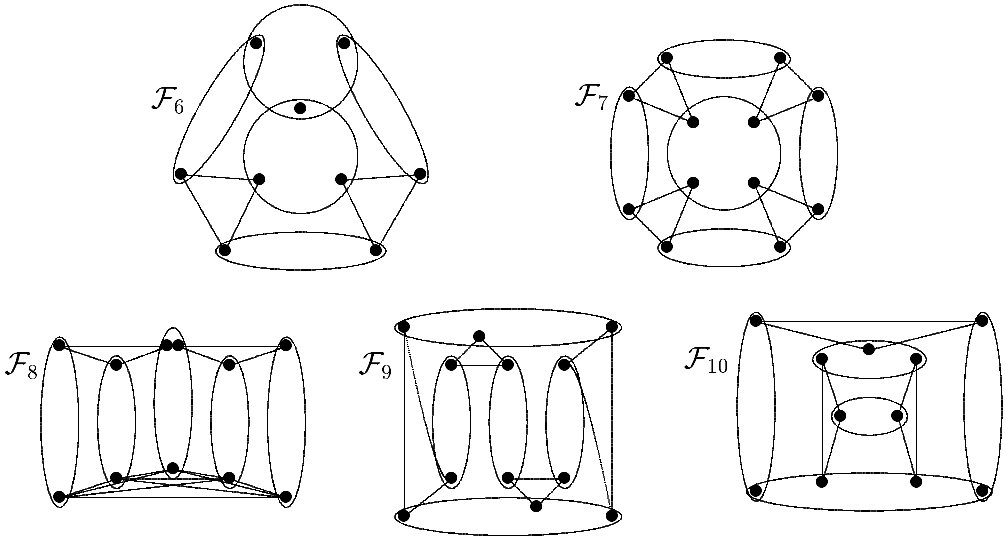


Figure 4.5

Similarly as in Theorem 4.13, all the nonhamiltonian exceptional graphs have connectivity 2 and hence, under the assumptions of Corollary 4.16, 3-connectedness implies hamiltonicity.

The search for the exception classes was continued in [51] for $\theta(G) = 6, 7$ with the help of a cluster of parallel computers. In this way, the following results were obtained (for the lists of exceptions see [51]).

k	Assumptions	Number of exception classes
7	$n \geq 112$, $\delta(G) \geq 17$, $\sigma_7(G) > n + 28$	42
8	$n \geq 153$, $\delta(G) \geq 20$, $\sigma_8(G) > n + 39$	318

In 3-connected claw-free graphs, Favaron and Fraisse [38] recently proved (using closure techniques) that every 3-connected claw-free graph with $\delta(G) \geq \frac{n+38}{10}$ is hamiltonian.

Kuipers and Veldman [52] further exploited the fact that the basic idea of finding the exceptional classes of Figure 4.5 yields a general method for listing these classes for any fixed upper bound on $\theta(G)$. This was a starting point for the proof of the following result. Consider the following two problems.

HAM(c)
Instance: A graph G with $\delta(G) \geq cn$.
Question: Is G hamiltonian?

HAMCL(c)
Instance: A claw-free graph G with $\delta(G) \geq cn$.
Question: Is G hamiltonian?

Häggkvist [40] proved that HAM($\frac{1}{2} - \varepsilon$) is NP-complete for any fixed $\varepsilon > 0$ (while HAM($\frac{1}{2}$) is trivial by Dirac's theorem). In claw-free graphs, hamiltonicity is known to be

NP-complete [6]. In contrast to these results, the result by Kuipers and Veldman [52] says that HAMCL(c) is polynomial for any $c > 0$.

Theorem 4.17 [52]. *HAMCL(c) is solvable in polynomial time for any constant $c > 0$.*

References

[1] Ainouche, A.; Christofides, N.: Strong sufficient conditions for the existence of hamiltonian cycles in undirected graphs . J. Combin. Theory B 31 (1981), 339-343.

[2] Ainouche, A.; Christofides, N.: Semi-independence number of a graph and the existence of hamiltonian circuits. Discrete Applied Math. 17 (1987), 213-221.

[3] Amar, D.; Favaron, O.; Mago, P.; Ordaz, O.: Biclosure and bistability in a balanced bipartite graph. J. Graph Theory 20 (1995), 513-529.

[4] Bauer, D.; Broersma, H.J.; Rao, L.; Veldman, H.J.: A generalization of a result of Häggkvist and Nicoghossian. J. Combin. Theory B 47 (1989), 237-243.

[5] Bedrossian, P.: Forbidden subgraph and minimum degree conditions for hamiltonicity. Thesis, Memphis State University, U.S.A., 1991.

[6] Bertossi, A.A.: The edge hamiltonian path problem is NP-complete. Inform. Process. Lett. 13 (1981), 157-159.

[7] Bollobás, B.; Riordan, O.; Ryjáček, Z.; Saito, A.; Schelp, R.H.: Closure and hamiltonian-connectivity of claw-free graphs. Discrete Math. 195 (1999), 171-177.

[8] Bondy, J.A.: Properties of graphs with constraints on degrees. Studia Sci. Math. Hungar. 4 (1969), 473-475.

[9] Bondy, J.A.; Chvátal, V.: A method in graph theory. Discrete Math. 15 (1976), 111-135.

[10] Bondy, J.A.; Murty, U.S.R.: Graph Theory with Applications (Macmillan, London and Elsevier, New York, 1976).

[11] Brandt, S.: Computing the independence number of dense triangle-free graphs. Graph-Theoretic Concepts in Computer Science (Proc. WG 97, R.H. Möhring, ed.), Lecture Notes in Computer Science 1335, Springer 1997, 100-108.

[12] Brandt, S.: 9-connected claw-free graphs are Hamilton-connected. J. Combin. Theory Ser. B 75 (1999), 167-173.

[13] Brandt, S.: On the structure of dense triangle-free graphs. Combinatorics, Probability and Computing 8 (1999), 237-246.

[14] Brandt, S.; Favaron, O.; Ryjáček, Z.: Closure and stable hamiltonian properties in claw-free graphs. J. Graph Theory (to appear).

[15] Brandt, S.; Veldman, H.J.: Degree sums for edges and cycle lengths in graphs. J. Graph Theory 25 (1997), 253-256.

[16] Broersma, H.J.: A note on hamiltonian graphs. Annals of Discrete Math. 41 (1989), 71-74.

[17] Broersma, H.J.: A note on K_4-closures in hamiltonian graph theory. Discrete Math. 121 (1993), 19-23.

[18] Broersma, H.J.; Schiermeyer, I.: A closure concept based on neighbourhood unions of independent triples. Discrete Math. 124 (1994), 37-47.

[19] Broersma, H.J.; Schiermeyer, I.: Subgraphs, closures and hamiltonicity. Discrete Applied Math. 51 (1994), 39-46.

[20] H.J. Broersma, H.J.; Trommel, H.: Closure concepts for claw-free graphs. Discrete Math. 185 (1998), 231-238.

[21] Brousek, J.: Minimal 2-connected non-hamiltonian claw-free graphs. Discrete Math. 191 (1998), 54-64.

[22] Brousek, J.; Favaron, O.; Ryjáček, Z.: Forbidden subgraphs, hamiltonicity and closure in claw-free graphs. Discrete Math. 196 (1999), 29-50.

[23] Brousek, J.; Ryjáček, Z.; Schiermeyer, I.: Forbidden subgraphs, stability and hamiltonicity. Discrete Math. 197/198 (1999), 143-155.

[24] Chvátal, V.: On hamilton's ideals. J. Combin. Theory B 12 (1972), 163-168.

[25] Chvátal, V.: Hamiltonian cycles, in: E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan and D.B. Shmoys, eds., The Traveling Salesman Problem (Wiley, Chichester, 1985), 403-429.

[26] Clark, L.; Entringer, R.C.; Jackson, D.E.: Minimum graphs with complete k-closure. Discrete Math. 30 (1980), 95-101.

[27] Dankelmann, P.; Niessen, T.; Schiermeyer, I.: On path-tough graphs. Siam Journal on Discrete Math. 7 (1994), 571-584.

[28] Dirac, G.A.: Some theorems on abstract graphs. Proc. London Math. Soc. 2 (1952), 69-81.

[29] Dirac, G.A.: Note on hamilton circuits and hamilton paths. Math. Ann. 206 (1973), 139-147.

[30] Fan, G: Personal communication (1997).

[31] Faudree, R.J.; Favaron, O.; Flandrin, E.; Li, H.: The complete closure of a graph. J. Graph Theory 17 (1993), 481-494.

[32] Faudree, R.J.; Favaron, O.; Flandrin, E.; Li, H.: Pancyclism and small cycles in graphs. Discussiones Mathematicae Graph Theory 16 (1996), 27-40.

[33] Faudree, R.J.; Flandrin, E.; Ryjáček, Z.: Claw-free graphs - a survey. Discrete Math. 164 (1997), 87-147.

[34] Faudree, R.J.; Gould, R.J.: Characterizing forbidden pairs for hamiltonian properties. Discrete Math. 173 (1997), 45-60.

[35] Faudree, R.J.; Gould, R.J.; Jacobson, M.S.; Lesniak, L.: Neighborhood closures of graphs. Colloq. Math. Soc. Janos Bolyai 52 (1987), 227-237.

[36] Faudree, R.J.; Gould, R.J.; Jacobson, M.S.; Schelp, R.H.: Neighborhood unions and hamiltonian properties of graphs. J. Combin. Theory B 47 (1989), 1-9.

[37] Favaron, O.; Flandrin, E.; Li, Hao; Ryjáček, Z.: Clique covering and degree conditions for hamiltonicity in claw-free graphs. Preprint 1997 (submitted).

[38] Favaron, O.; Fraisse, P.: Hamiltonicity and minimum degree in 3-connected claw-free graphs. Rapport de Recherche No. 1219, L.R.I., Univ. Paris-Sud, Orsay, France, 1999.

[39] Gurgel, M.A.; Wakabayashi, Y.: On k-leaf-connected graphs. J. Combin. Theory B 41 (1986), 1-16.

[40] Häggkvist, R.: On the structure of non-hamiltonian graphs I. Combinatorics, Probability and Computing 1 (1992), 27-34.

[41] Hanson, D.: Hamilton closures in domination critical graphs. J. Comb. Math. and Comb. Computing 13 (1993), 121-128.

[42] Hasratian, A.S.; Khachatrian, N.K.: Some localization theorems on hamiltonian circuits. J. Combin. Theory B 49 (1990), 287-294.

[43] Hasratian, A.S.; Khachatrian, N.K.: Stable properties of graphs. Discrete Math. 90 (1991), 143-152.

[44] Hayes, D.; Schmeichel, E.: Some extensions of Ore's theorem, in: Graph Theory and its Applications to Algorithms and Computer Science (Wiley, New York, 1985), 687-695.

[45] Hendry, R:T.: Extending cycles in graphs. Discrete Math. 85 (1990), 59-72.

[46] Ishizuka, S.: Closure, path-factors and path coverings in claw-free graphs. Ars Combinatoria 50 (1998), 115-128.

[47] Ishizuka, S.: Closure and $(2, n)$-factor-criticality of claw-free graphs. Preprint 1998, Keio University, Yokohama, Japan.

[48] Jackson, B.: Hamilton cycles in 7-connected line graphs. Preprint 1989 (unpublished).

[49] Jung, H.A.: On maximal circuits in finite graphs. Annals of Discrete Math. 5 (1978), 129-144.

[50] Khuller, S.: On computing graph closures. Information Processing Letters 31 (1989), 249-255.

[51] Kovářík, O.; Mulač, M.; Ryjáček, Z.: A note on degree conditions for hamiltonicity in 2-connected claw-free graphs. Preprint No. 124/1999, Univ. of West Bohemia, Pilsen, 1999 (submitted).

[52] Kuipers, E.J.; Veldman, H.J.: Recognizing claw-free hamiltonian graphs with large minimum degree. Preprint 1998, (submitted).

[53] Las Vergnas, M.: Thesis, University of Paris VI, France, 1972.

[54] Lesniak, L.: Some recent results in hamiltonian graphs. J. Graph Theory 1 (1977), 27-36.

[55] Li, H: A note on hamiltonian claw-free graphs. Rapport de Recherche 1022, Univ. Paris-Sud, Orsay, France, 1996.

[56] Möhring, R.H.: Triangulating graphs without asteroidal triples. Discrete Applied Math. 64 (1996), 281-287.

[57] Monti, A.: On the computational complexity of graph closures. Information Processing Letters 57 (1996), 291-295.

[58] Niessen, T.: Complete closure and regular factors. J. Comb. Math. and Comb. Computing 25 (1997), 145-160.

[59] Niessen, T.: Personal communication (1998).

[60] Ore, O.: Note on hamiltonian circuits. Amer. Math. Monthly 67 (1960), 55.

[61] Plummer, M.D.; Saito, A.: Closure and factor-critical graphs. Preprint 1997.

[62] Pósa, L.: A theorem concerning hamilton lines. Magyar Tud. Akad. Mat. Kukato Int. Közl. 8 (1963), 335-361.

[63] Randerath, B.; Schiermeyer, I.; Tewes, M.; Volkmann, L.: Vertex pancyclic graphs. Manuscript 1998.

[64] Ryjáček, Z.: On a closure concept in claw-free graphs. J. Combin. Theory B 70 (1997), 217-224.

[65] Ryjáček, Z.; Saito, A.; Schelp, R.H.: Closure, 2-factors and cycle coverings in claw-free graphs. J. Graph Theory (to appear).

[66] Ryjáček, Z.; Saito, A.; Schelp, R.H.: Claw-free graphs with complete closure. Preprint 1997 (submitted).

[67] Saito, A.; Schiermeyer, I.: Private communication, 1999.

[68] Schelten, U.: Knotengrade und Nachbarschaftsvoraussetzungen als hinreichende Bedingungen für Kreise in Graphen. Dissertation, RWTH Aachen, Germany, 1997.

[69] Schelten, U.; Schiermeyer, I.: Small cycles in hamiltonian graphs. Discrete Applied Math. 79 (1997), 201-211.

[70] Schiermeyer, I.: Algorithmen zum Auffinden längster Kreise in Graphen. Dissertation, RWTH Aachen, Germany, 1988.

[71] Schiermeyer, I.: A fast sequential and parallel algorithm for the computation of the k-closure of a graph. WG'89, Lecture Notes in Computer Science 411 (1989), 211-217.

[72] Schiermeyer, I.: Triple closure operations and graphical properties. RWTH Aachen, Preprint 1990.

[73] Schiermeyer, I.: Computation of the 0-dual closure for several classes of hamiltonian graphs. Discrete Math. 111 (1993), 455-464.

[74] Skupień, Z.: n-Closures of large nonhamiltonian n-vertex graphs. In: Graphs, Hypergraphs and Matroids. Zielona Góra (1985), 59-66.

[75] Stacho, L.: Degrees, neighborhoods and closure operations. Preprint 1999.

[76] Szwarcfiter, J.L.: A note on the computation of the k-closure of a graph. Information Processing Letters 24 (1987), 279-280.

[77] Zhan, S.: On hamiltonian line graphs and connectivity. Discrete Math. 89 (1991), 89-95.

[78] Zhu, Y.; Tian, F.: A generalization of the Bondy-Chvátal Theorem on the k-closure. J. Combin. Theory B 35 (1983), 247-255.

[79] Zhu, Y.; Tian, F.; Deng, X.: More powerful closure-operations on graphs. Discrete Math. 87 (1991), 197-214.