Strengthening the closure concept in claw-free graphs

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Abstract

We give a strengthening of the closure concept for claw-free graphs introduced by the second author in 1997. The new closure of a claw-free graph G defined here is uniquely determined and preserves the value of the circumference of G. We present an infinite family of graphs with n vertices and $\frac{3}{2}n - 1$ edges for which the new closure is the complete graph K_n.

Keywords: closure, cycle closure, claw-free graph, circumference, hamiltonian graph

AMS Subject Classifications (1991): 05C45, 05C35

1 Introduction

We consider finite simple undirected graphs $G = (V(G), E(G))$. For concepts and notation not defined here we refer the reader to [1]. We denote by $c(G)$ the *circumference* of G, i.e. the length of a longest cycle in G, by $N_G(x)$ the *neighborhood* of a vertex x in G (i.e., $N_G(x) = \{y \in V(G)|\ xy \in E(G)\}$), and we denote $N_G[x] = N_G(x) \cup \{x\}$. For a nonempty set $A \subseteq V(G)$, the *induced subgraph on* A is denoted by $\langle A \rangle_G$, the notation $G - A$ stands for $\langle V(G) \setminus A \rangle_G$ (if $A \neq V(G)$) and we put $N_G(A) = \{x \in V(G)|\ N(x) \cap A \neq \emptyset\}$ and $N_G[A] = N_G(A) \cup A$. For a subgraph X of G we denote $N_G(X) = N_G(V(X))$ and $N_G[X] = N_G[V(X)]$.

If F is a graph, then we say that a graph G is F-free if G does not contain a copy of F as an induced subgraph. The graph $K_{1,3}$ will be called the *claw* and in the special case $F = K_{1,3}$ we say that G is *claw-free* (instead of F-free). The *line graph* of a graph H is denoted by $L(H)$. If $G = L(H)$, then we also say that H is the *line graph preimage* of G and denote $H = L^{-1}(G)$. It is well-known that for any connected line graph $G \not\simeq K_3$ its line graph preimage is uniquely determined.

Let T be a closed trail in G. We say that T is a *dominating closed trail* (abbreviated DCT), if $V(G) \setminus V(T)$ is an independent set in G (or, equivalently, if every edge of G has at least one vertex on T). Harary and Nash-Williams [6] proved the following result, relating the existence of a DCT in a graph to the hamiltonicity of its line graph.

Theorem A [6]. *Let H be a graph with $|E(H)| \geq 3$ without isolated vertices. Then $L(H)$ is hamiltonian if and only if H contains a DCT.*

A special case is that $H = K_{1,r}$ for some $r \geq 3$; then $L(H) = K_r$ and the DCT in H consists of a single vertex.

For a vertex $x \in V(G)$, set $B_x = \{uv|\ u, v \in N(x), uv \notin E(G)\}$ and $G'_x = (V(G), E(G) \cup B_x)$. The graph G'_x is called the *local completion of* G *at* x. It was proved in [8] that if G is claw-free, then so is G'_x, and if $x \in V(G)$ is a *locally connected vertex* (i.e., $\langle N(x) \rangle_G$ is a connected graph), then $c(G) = c(G'_x)$. A locally connected vertex x with $B_x \neq \emptyset$ is called *eligible* (in G) and the set of all eligible vertices of G is denoted by $V_{EL}(G)$.

We say that a graph F is a *closure of* G, denoted $F = \mathrm{cl}(G)$ (see [8]), if $V_{EL}(F) = \emptyset$ and there is a sequence of graphs $G_1, \ldots, G_t$ and vertices $x_1, \ldots, x_{t-1}$ such that $G_1 = G$, $G_t = F$, $x_i \in V_{EL}(G_i)$ and $G_{i+1} = (G_i)'_{x_i}$, $i = 1, \ldots, t-1$ (equivalently, $\mathrm{cl}(G)$ is obtained from G by a series of local completions at eligible vertices, as long as this is possible). The following basic result was proved in [8].

Theorem B [8]. *Let G be a claw-free graph. Then*

(i) $\mathrm{cl}(G)$ *is well-defined (i.e., uniquely determined),*

(ii) *there is a triangle-free graph H such that* $\mathrm{cl}(G) = L(H)$,

(iii) $c(G) = c(\mathrm{cl}(G))$.

Consequently, a claw-free graph G is hamiltonian if and only if so is its closure $\mathrm{cl}(G)$. A claw-free graph G for which $G = \mathrm{cl}(G)$ will be called *closed*. Clearly, G is closed if and only if $V_{EL}(G) = \emptyset$, i.e. if every vertex $x \in V(G)$ is either *simplicial* ($\langle N(x)\rangle_G$ is a clique), or is *locally disconnected* ($\langle N(x)\rangle_G$ is disconnected, implying that, since G is claw-free, $\langle N(x)\rangle_G$ consists of two vertex disjoint cliques). It is easy to observe that G is a closed claw-free graph if and only if G is claw-free and $(K_4 - e)$-free. This implies that if G is closed claw-free, then so is every induced subgraph of G. It is also straightforward to check that for any edge e of a closed claw-free graph the largest clique containing e is uniquely determined. The order of the largest clique in a closed claw-free graph G containing a given edge e will be denoted by $\omega_G(e)$.

The closure concept for claw-free graphs has been studied intensively since it has been introduced in [8]. It is known to preserve a number of graph properties and values of graph parameters, and has found many applications. Interested readers can find more information e.g. in the survey paper [3].

In the following section we introduce a strengthening of this closure concept, and we show that this new closure is again uniquely determined and that it preserves the value of the circumference of G.

2 The cycle closure

Let G be a closed claw-free graph and let C be an induced cycle in G of length k. We say that the cycle C is *eligible in* G if $4 \leq k \leq 6$ and $\omega_G(e) = 2$ for at least $k-3$ nonconsecutive edges $e \in E(C)$ (or, equivalently, if the k-cycle $L^{-1}(C)$ in $H = L^{-1}(G)$ contains at least $k-3$ nonconsecutive vertices of degree 2).

For an eligible cycle C in G set $B_C = \{uv \mid u, v \in N_G[C], uv \notin E(G)\}$. The graph G'_C with vertex set $V(G'_C) = V(G)$ and edge set $E(G'_C) = E(G) \cup B_C$ is called the *C-completion* of G at C.

The following proposition shows that the C-completion of a closed claw-free graph at an eligible cycle C is again claw-free and has the same circumference. Note that a C-completion of a closed claw-free graph is not necessarily closed (for example, the graph G with $V(G) = \{a, b, c, d, e, f, g\}$ and $E(G) = \{ab, bc, cd, de, ef, fa, ga, gb, gd, ge\}$ is closed and claw-free, the 4-cycle $C = agefa$ is eligible in G, but G'_C is not closed since $b, d \in V_{EL}(G'_C)$).

Proposition 1. *Let G be a closed claw-free graph, let C be an eligible cycle in G and let G'_C be the C-completion of G. Then*

(i) G'_C is claw-free,

(ii) $c(G'_C) = c(G)$.

Proof. (i) Let $H = \langle\{z, y_1, y_2, y_3\}\rangle_{G'_C}$ be a claw. Then $1 \leq |E(H) \cap B_C|$ since G is claw-free, and $|E(H) \cap B_C| \leq 1$ since $\langle N[C]\rangle_{G'_C}$ is a clique. Let $zy_1 \in B_C$. Then $z \in N[C]$, implying $zu \in E(G)$ for some $u \in V(C)$. Then obviously $uy_2, uy_3 \notin E(G)$ (otherwise H is not a claw in G'_C), but then $\langle\{z, u, y_2, y_2\}\rangle_G$ is a claw in G, a contradiction.

(ii) Obviously $c(G'_C) \geq c(G)$ since every cycle in G is a cycle in G'_C. To prove the converse, it is sufficient to show that for every longest cycle C'_1 in G'_C there is a cycle C_1 in G with $V(C_1) = V(C'_1)$. This is clear if $E(C'_1) \cap B_C = \emptyset$; hence suppose $E(C'_1) \cap B_C \neq \emptyset$. Since C'_1 is longest and $\langle N[C] \rangle_{G'_C}$ is a clique, $N[C] \subset V(C'_1)$, implying that $\langle V(C'_1) \rangle_{G'_C}$ is the C-completion of $\langle V(C'_1) \rangle_G$. Since every induced subgraph of a closed claw-free graph is again claw-free and closed, it is sufficient to show that if G'_C is hamiltonian then so is G.

Let $H = L^{-1}(G)$ and suppose that C is a k-cycle ($4 \leq k \leq 6$). Since C is eligible in G, the k-cycle $L^{-1}(C)$ in H contains $k-3$ nonconsecutive vertices x_i, $i = 1, \ldots, k-3$, of degree 2. Let x_i^-, x_i^+ be the predecessor and successor of x_i on $L^{-1}(C)$, respectively.

It is straightforward to check that G'_C can be equivalently obtained by the following construction:

- (i) denote by H' the graph obtained from H by replacing the path $x_i^- x_i x_i^+$ by the edge $x_i^- x_i^+$, $i = 1, \ldots, k-3$;
- (ii) denote by a_i the vertices of $L(H')$ corresponding to the edges $x_i^- x_i^+$, $i = 1, \ldots, k-3$;
- (iii) construct a graph $\bar{G}$ from $L(H')$ by a series of consecutive local completions at the vertices $a_1, \ldots, a_{k-3}$;
- (iv) add $k-3$ vertices $z_1, \ldots, z_{k-3}$ to $\bar{G}$ and turn the set $\{z_1, \ldots, z_{k-3}\} \cup N_{\bar{G}}[\{a_1, \ldots, a_{k-3}\}]$ into a clique.

Note that step (i) turns C into a triangle, and hence the vertices $a_1, \ldots, a_{k-3}$ are locally connected in $L(H')$.

By the main result of [8], by the above considerations and by Theorem A, it is sufficient to show that if H' contains a DCT, then so does H. Let T be a DCT in H'.

Suppose first that $k = 4$ and, for simplicity, set $x = x_1$. If $x^- x^+ \in E(T)$, then, replacing in T the edge $x^- x^+$ by the path $x^- x x^+$, we have a DCT in H. Hence suppose $x^- x^+ \notin E(T)$. Since T is dominating, $|\{x^-, x^+\} \cap V(T)| \geq 1$. If both x^-, x^+ are on T, then T is dominating in H. Hence we can suppose $x^- \in V(T)$ and $x^+ \notin V(T)$. If $x^- x^{++} \in E(T)$, then we replace in T the edge $x^- x^{++}$ by the path $x^- x x^+ x^{++}$, and if $x^- x^{++} \notin E(T)$, then we add to T the 4-cycle $x^- x x^+ x^{++} x^-$. In both cases, we have a DCT in H.

Let now $k = 5$ and suppose the notation is chosen such that $x_1^+ = x_2^-$. If $x_1^- x_1^+ \in E(T)$ and $x_2^- x_2^+ \in E(T)$, then, replacing in T the edges $x_1^- x_1^+$ and $x_2^- x_2^+$ by the paths $x_1^- x_1 x_1^+$ and $x_2^- x_2 x_2^+$, we have a DCT in H. If $x_1^- x_1^+ \notin E(T)$ and $x_2^- x_2^+ \notin E(T)$, then for $x_1^- x_2^+ \in E(T)$ we replace in T the edge $x_1^- x_2^+$ by the path $x_1^- x_1 x_1^+ x_2 x_2^+$, and for $x_1^- x_2^+ \notin E(T)$ we add to T the cycle $x_1^- x_1 x_1^+ x_2 x_2^+ x_1^-$. In both cases, we have a DCT in H (note that at least two of the vertices x_1^-, x_1^+, x_2^+ are on T since T is dominating). Up to symmetry, it remains to consider the case when $x_1^- x_1^+ \in E(T)$ and $x_2^- x_2^+ \notin E(T)$. Then for $x_1^- x_2^+ \in E(T)$ the trail T is a DCT in H, and for $x_1^- x_2^+ \notin E(T)$ we get a DCT in H by replacing in T the edge $x_1^- x_1^+$ by the path $x_1^- x_2^+ x_2 x_2^- (= x_1^+)$. Thus, in all cases we have a DCT in H.

Finally, let $k = 6$ and choose the notation such that $x_1^+ = x_2^-$ and $x_2^+ = x_3^-$. If at least two of the edges $x_1^+ x_2^+$, $x_2^+ x_3^+$, $x_3^+ x_1^+$ are on T (say, $x_1^+ x_2^+$, $x_2^+ x_3^+$ are on T), then, replacing in T the edges $x_1^+ x_2^+$ and $x_2^+ x_3^+$ by the paths $x_1^+ x_2 x_2^+$ and $x_2^+ x_3 x_3^+$, we get a DCT in H. If none of the edges $x_1^+ x_2^+$, $x_2^+ x_3^+$, $x_3^+ x_1^+$ is on T, then we get a DCT in H by adding to T the cycle $x_1x_1^+x_2x_2^+x_3x_3^+x_1$ (note that again at least two of the vertices x_1^+, x_2^+, x_3^+ are on T since T is dominating). Hence it remains to consider the case that exactly one of these edges, say, $x_1^+x_2^+$, is on T, but in this case we obtain a DCT in H by replacing in T the edge $x_1^+x_2^+$ by the path $x_1^+x_1x_3^+x_3x_2^+$. ■

Now we can define the main concept of this paper which strengthens the closure concept introduced in [8].

Definition 2. *Let G be a claw-free graph. We say that a graph F is a* cycle closure *of G, denoted $F = \mathrm{cl}_C(G)$, if there is a sequence of graphs $G_1, \ldots, G_t$ such that*
(i) $G_1 = \mathrm{cl}(G)$,
(ii) $G_{i+1} = \mathrm{cl}((G_i)'_C)$ for some eligible cycle C in G_i, $i = 1, \ldots, t-1$,
(iii) $G_t = F$ contains no eligible cycle.

Thus, $\mathrm{cl}_C(G)$ is obtained from $\mathrm{cl}(G)$ by recursively performing C-completion operations at eligible cycles and each time closing the resulting graphs with the closure defined in [8], as long as this is possible (i.e., as long as there is some eligible cycle). It is easy to see that $\mathrm{cl}_C(G)$ can be computed in polynomial time.

It follows immediately from the definition that $E(\mathrm{cl}(G)) \subseteq E(\mathrm{cl}_C(G))$ for any claw-free graph G. We show that $\mathrm{cl}_C(G)$ is well-defined (i.e., uniquely determined) and that the cycle closure operation preserves the value of the circumference of G.

Theorem 3. *Let G be a claw-free graph. Then*
(i) $\mathrm{cl}_C(G)$ is well-defined,
(ii) $c(G) = c(\mathrm{cl}_C(G))$.

From Theorem 3 we immediately have the following consequence.

Corollary 4. *Let G be a claw-free graph. Then*
(i) G is hamiltonian if and only if $\mathrm{cl}_C(G)$ is hamiltonian;
(ii) if $\mathrm{cl}_C(G)$ is complete, then G is hamiltonian.

Before proving Theorem 3, we first prove the following lemma.

Lemma 5. *Let G be a closed claw-free graph, let C, C_1 be two eligible cycles in G and let $G' = \mathrm{cl}(G'_C)$, where G'_C is the C-completion of G at C. Then either $\langle V(C_1)\rangle_{G'}$ is a clique, or there is a cycle C_2 such that $V(C_2) \subseteq V(C_1)$, C_2 is eligible in G' and, in the graph $G'' = (G')'_{C_2}$, $\langle V(C_1)\rangle_{G''}$ is a clique.*

This implies, in particular, that all vertices of C_1 are locally connected in G' or G'', respectively.

Proof. The last statement follows obviously from the eligibility of C_1 in G and the completeness of $\langle V(C_1)\rangle_{G'}$ or $\langle V(C_1)\rangle_{G''}$, respectively. To prove the first statement, denote by $k = |V(C_1)|$ and let $e_i = a_i a_i^+$ $(i = 1, \ldots, k-3)$ be the nonconsecutive edges of C_1 with $\omega_G(e_i) = 2$. Suppose the notation is chosen such that $a_1^+ = a_2^-$ if $k \geq 5$ and, moreover, $a_2^+ = a_3^-$ if $k = 6$. We can suppose that $\langle V(C_1)\rangle_{G'}$ is not a clique (otherwise we are done) and that C_1 is not eligible in G' (otherwise we are done with $C_2 = C_1$).

Suppose that $\omega_{G'}(e_i) = 2$ for all i, $1 \leq i \leq k-3$. Since C_1 is not eligible, C_1 is not an induced cycle in G'. For $k = 4$ this immediately implies that $\langle V(C_1)\rangle_{G'}$ is a clique (since G is closed), a contradiction. For $k = 5$, the only chord in C_1 is $a_1 a_2^+$ (all other chords would imply $\omega_{G'}(e_i) \geq 3$ for some i), but then we are done with $C_2 = a_1 a_1^+ a_2 a_2^+ a_1$. For $k = 6$, any chord in C_1 implies $\omega_{G'}(e_i) \geq 3$ for some i (using the fact that G' is claw-free). Hence we can suppose that $\omega_{G'}(e_i) \geq 3$ for some i, $1 \leq i \leq k-3$. By symmetry, suppose that $\omega_{G'}(e_1) \geq 3$. We claim the following.

Claim 1. Let $e = aa^+$ be an edge of C_1 such that $\omega_G(e) = \omega_{G'_C}(e) = 2$ but $\omega_{G'}(e) \geq 3$. Then either $aa^{++} \in E(G')$, or $a^- a^+ \in E(G')$.

Proof of Claim 1. Suppose that $\omega_{G'}(e) \geq 3$. By the definition of G', there is a sequence of graphs $F_1, \ldots, F_\ell$ and vertices $x_1, \ldots, x_{\ell-1}$ such that $F_1 = G'_C$, $F_\ell = G'$, $x_1 \in V_{EL}(F_i)$ and $F_{i+1} = (F_i)'_{x_i}$, $i = 1, \ldots, \ell-1$. Let j $(1 \leq j \leq \ell-1)$ be the smallest integer for which $\omega_{F_j}(e) \geq 3$. Then there is a vertex $c \in V(G)$ such that $ca, ca^+ \in E(F_j)$, but at least one of ca, ca^+ is not in $E(F_{j-1})$.

Let first $ca \notin E(F_{j-1})$. Then $cx_{j-1}, ax_{j-1} \in E(F_{j-1})$. Clearly $x_{j-1}a^+ \notin E(F_{j-1})$ (otherwise $\omega_{F_{j-1}}(e) \geq 3$) and $a^- a^+ \notin E(F_{j-1})$ (otherwise there is nothing to prove). Since $\langle\{a, a^-, a^+, x_{j-1}\}\rangle_{F_{j-1}}$ is not a claw, we have $x_{j-1}a^- \in E(F_{j-1})$. From $x_{j-1}a^+ \notin E(F_{j-1})$ we also have $ca^+ \in E(F_{j-1})$, since otherwise cannot be $ca^+ \in E(F_j)$. But then $a^+ c x_{j-1} a^-$ is an (a^+, a^-)-path in $N_{F_j}(a)$, implying $a \in V_{EL}(F_j)$, from which, since $G' = \mathrm{cl}(F_j)$, we have $a^- a^+ \in E(G')$.

If $ca^+ \notin E(F_{j-1})$, then symmetrically $aa^{++} \in E(G')$. Hence the claim follows. $\square$

Claim 2. Let $e = aa^+$ be an edge of C_1 such that $\omega_G(e) = 2$ and $\omega_{G'_C}(e) \geq 3$. Then $\langle\{a^-, a, a^+, a^{++}\}\rangle_{G'}$ is a clique.

Proof of Claim 2. Let $c \in V(G)$ be such that $ca, ca^+ \in E(G'_C)$. By symmetry, suppose $ca^+ \notin E(G)$. Then $c, a^+ \in N_G[C]$. Let d be a neighbor of a^+ on C, and denote by K^+ (K^-) the largest clique in G, containing the edge $a^+ a^{++}$ $(a^- a)$, respectively. Since $\langle\{a^+, a^{++}, a, d\}\rangle_G$ cannot be a claw and $da, a^{++}a \notin E(G)$ (since $\omega_G(e) = 2$), we have $da^{++} \in E(G)$, implying, since G is closed, $d \in V(K^+)$. Since $cd, ca^+ \in E(G'_C)$ and G' is closed, we have $aa^{++} \in E(G')$. For $k = 4$ this immediately implies that $\langle V(C_1)\rangle_{G'}$ is a clique, hence $|V(C_1)| \geq 5$.

Now we consider the edge ca. If $ca \notin E(G)$, then, by a symmetric argument, we have $a^- a^+ \in E(G')$ and we are done since G' is closed. Hence $ca \in E(G)$. Since $\langle\{a, c, a^+, a^-\}\rangle_G$ cannot be a claw and $ca^+ \notin E(G)$, either $a^- a^+ \in E(G)$ (and we are done), or $ca^- \in E(G)$, implying $c \in V(K^-)$. But then, since $ca^+ \in E(G'_C)$ and G' is closed, again $a^- a^+ \in E(G')$ and hence also $a^- a^{++} \in E(G')$. This proves Claim 2. $\square$

Now for $k=4$ from $\omega_{G'}(e_1)\geq 3$ and from Claims 1 and 2 we immediately have that $\langle V(C_1)\rangle_{G'}$ is a clique.

Let $k=5$. If $\omega_{G'_C}(e_1)\geq 3$, then $\langle V(C_1)\rangle_{G'}$ is a clique by Claim 2 and since G' is closed. Thus, let $\omega_{G'_C}(e_1)=2$. By Claim 1, $a_1^-a_1^+\in E(G')$ or $a_1a_2\in E(G')$. If both these edges are present or if $\omega_{G'}(e_2)\geq 3$, then clearly $\langle V(C_1)\rangle_{G'}$ is a clique. Otherwise, we set $C_2=a_1^-a_1^+a_2a_2^+a_1^-$ (if $a_1^-a_1^+\in E(G')$) or $C_2=a_1a_2a_2^+a_1^-a_1$ (if $a_1a_2\in E(G')$).

Finally, suppose that $k=6$. We show that $\omega_{G'_C}(e_1)=2$. If $\omega_{G'_C}(e_1)\geq 3$ and $\omega_{G'}(e_2)\geq 3$ or $\omega_{G'}(e_3)\geq 3$, then, by Claims 1 and 2 and since G' is closed, $\langle V(C_1)\rangle_{G'}$ is a clique. If $\omega_{G'_C}(e_1)\geq 3$ and $\omega_{G'}(e_2)=\omega_{G'}(e_3)=2$, then we are done with $C_2=a_2a_2^+a_3a_3^+a_2$. Hence $\omega_{G'_C}(e_1)=2$. By a symmetric argument we can prove that also $\omega_{G'_C}(e_2)=\omega_{G'_C}(e_3)=2$. By the assumption $\omega_{G'}(e_1)\geq 3$ and by Claim 1, at least one of the chords $a_1^-a_1^+$, a_1a_2 is present. Now, if both $a_2^-a_2^+\in E(G')$ and $a_2a_3\in E(G')$, then, since G' is closed, also $a_2^-a_3\in E(G')$, which together with any of the chords $a_1^-a_1^+$, a_1a_2 implies that $\langle V(C_1)\rangle_{G'}$ is a clique. Hence at most one of $a_2^-a_2^+$, a_2a_3 is present. Symmetrically, at most one of $a_3^-a_3^+$, a_3a_1 is present. Hence we have at least one of the chords $a_1^-a_1^+$, a_1a_2, at most one of $a_2^-a_2^+$, a_2a_3, and at most one of $a_3^-a_3^+$, a_3a_1. Then it is straightforward to check that in each of the possible cases either $\langle V(C_1)\rangle_{G'}$ is a clique or we can find a required cycle C_2. ∎

Proof of Theorem 3. (i) Let F_1, F_2 be two cycle closures of G, suppose $E(F_1)\setminus E(F_2)\neq\emptyset$ and let $G_1,\ldots,G_t$ be the sequence of graphs that yields F_1. Let $e=xy\in E(G_j)\setminus E(F_2)$ be chosen such that j is as small as possible. Since $e\in E(G_j)$, either $x,y\in N[C]$ for some eligible cycle C in G_{j-1}, or there is a sequence of vertices $x_1,\ldots,x_k$ and graphs $H_1,\ldots,H_k$ such that $H_1=(G_{j-1})'_C$, x_i is eligible in H_i, $H_{i+1}=(H_i)'_{x_i}$, $i=1,\ldots,k$, and $x,y\in N_{H_k}(x_k)$. By Lemma 5 (in the first case) and since obviously a locally connected vertex remains locally connected after adding edges to the graph (in the second case), we have $xy\in E(F_2)$, a contradiction.

(ii) Part (ii) follows immediately from Proposition 1 and from the main result of [8]. ∎

Example 1. The graph in Figure 1a) shows that Proposition 1 fails if we require only one edge e with $\omega_G(e)=2$ in a C_5 or if we admit the two edges to be consecutive. The graph in Figure 1b) gives a similar example for a C_6 (elliptical parts represent cliques of order at least three).

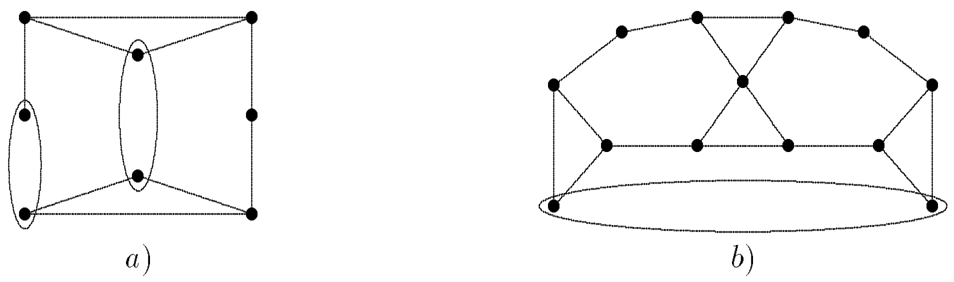

Figure 1

Example 2. Linderman [7] proved that the minimum number of edges of a claw-free graph G of order n with a complete closure $\mathrm{cl}(G)$ equals $2n-3$. The graph in Figure 2 is an example of a claw-free graph G of order $n \equiv 0 \pmod 6$ with a complete cycle closure $\mathrm{cl}_C(G)$ and with only $\frac{3}{2}n - 1$ edges.

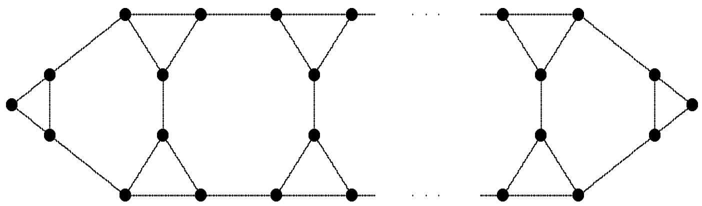

Figure 2

Remarks. (i) The graph in Figure 2 is a closed claw-free graph that contains neither a C_4 nor a $K_4 - e$ as an induced subgraph. This implies that the closure concepts based on neighborhood conditions for the vertices of an induced $K_4 - e$ introduced in [2] and [4] cannot be applied to add new edges to this graph (while its cycle closure is a complete graph). On the other hand, the closures from [2] and [4] do not assume claw-freeness of the original graph, and yield additional edges in graphs for which the closure of [8] and the cycle closure are not defined.

(ii) Catlin [5] has introduced a powerful reduction technique that reduces the order of the line graph preimage, preserving the existence of a spanning closed trail, and, with some restrictions, of a DCT in this preimage. Considering the graph $H = K_{2,t}$ for $t \geq 3$, it is not difficult to check that H is equal to its reduction (i.e. Catlin's reduction technique is not applicable), $L(H)$ is a closed claw-free graph (hence the closure technique introduced in [8] is also not applicable), but the cycle closure of $L(H)$ is a complete graph. This example shows that the cycle closure technique is not a special case of Catlin's reduction technique. Moreover, it is not known whether the reduction of a graph in the sense of Catlin's technique can be obtained in polynomial time. The same holds for the refinement of Catlin's technique due to Veldman [10].

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