A note on degree conditions for hamiltonicity in 2-connected claw-free graphs

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June 21, 2000

Abstract

Let G be a claw-free graph and let $\mathrm{cl}(G)$ be the closure of G. We present a method for characterizing classes $\mathcal{G}_i$, $i = 3, \ldots, 7$, of 2-connected closed claw-free graphs with the following properties.

(i) **Theorem.** Let G be a 2-connected claw-free graph of order $n \geq 153$ such that $\delta(G) \geq 20$ and $\sigma_8(G) > n + 39$. Then either G is hamiltonian or $\mathrm{cl}(G) \in \bigcup_{i=3}^{7} \mathcal{G}_i$.

(ii) **Corollary.** Let G be a 2-connected claw-free graph of order $n \geq 153$ with $\delta(G) \geq \frac{n+39}{8}$. Then either G is hamiltonian or $\mathrm{cl}(G) \in \bigcup_{i=3}^{7} \mathcal{G}_i$.

The family of exceptions contains 318 infinite classes. The majority of these exception classes were found with the help of a computer.

Keywords: hamiltonicity, claw, degree condition, 2-connected, closure

1991 Mathematics Subject Classification: 05C45, 05C70.

*Research supported by grant MŠMT ČR Infra II No. LB 98246

1 Introduction

We consider finite undirected graphs $G = (V(G), E(G))$ without loops and multiple edges. We follow the most common terminology and notation and for concepts not defined here we refer e.g. to [1]. For any set $A \subset V(G)$ we denote by $\langle A \rangle_G$ the subgraph of G induced on A and $G - A$ stands for $\langle V(G) \setminus A \rangle$. A graph G is *H-free* (where H is a graph), if G does not contain an induced subgraph isomorphic to H. In the special case $H = K_{1,3}$ we say that G is *claw-free*. The *independence number* of G is denoted by $\alpha(G)$ and the *clique covering number* of G (i.e. the minimum number of cliques necessary for covering $V(G)$) by $\theta(G)$. For a set $Y \subset V(G)$, $G|_Y$ is the graph obtained by contracting $\langle Y \rangle_G$ to a vertex, i.e. the graph with vertex set $V(G|_Y) = (V(G) \setminus Y) \cup \{x\}$ (where $x \notin V(G)$) and edge set $E(G|_Y) = E(G - Y) \cup \{wx \mid w \in V(G) \setminus Y \text{ and } wz \in E(G) \text{ for some } z \in Y\}$. We denote by $\delta(G)$ the *minimum degree* of G and by $\sigma_k(G)$ $(k \geq 1)$ the *minimum degree sum* over all independent sets of k vertices in G (for $k > \alpha(G)$ we set $\sigma_k(G) = \infty$).

The *line graph* of a graph H is denoted by $L(H)$. If $G = L(H)$, then we also denote $H = L^{-1}(G)$ and say that H is the *line graph preimage* of G (recall that for any line graph G nonisomorphic to K_3, its line graph preimage is uniquely determined).

A vertex $x \in V(G)$ is said to be *locally connected* if its neighborhood $N(x)$ induces a connected graph. The *closure* of a claw-free graph G (introduced in [11] by the third author) is defined as follows: the closure $\mathrm{cl}(G)$ of G is the (unique) graph obtained by recursively completing the neighborhood of any locally connected vertex of G, as long as this is possible. The closure $\mathrm{cl}(G)$ remains a claw-free graph and its connectivity is at least equal to the connectivity of G. The following basic properties of the closure $\mathrm{cl}(G)$ were proved in [11].

Theorem A [11]. *Let G be a claw-free graph and $\mathrm{cl}(G)$ its closure. Then*

(*i*) *there is a triangle-free graph H_G such that $\mathrm{cl}(G) = L(H_G)$,*

(*ii*) *the length of a longest cycle in G and in $\mathrm{cl}(G)$) is the same.*

Consequently, G is hamiltonian if and only if $\mathrm{cl}(G)$ is hamiltonian. If G is a claw-free graph such that $G = \mathrm{cl}(G)$, then we say that G is *closed*. It is apparent that a claw-free graph G is closed if and only if every vertex $x \in V(G)$ is either *simplicial* (i.e. $\langle N(x) \rangle_G$ is a clique), or is *locally disconnected* (i.e. $\langle N(x) \rangle_G$ consists of two vertex disjoint cliques).

A closed trail T in a graph H is said to be *dominating* if every edge of H has at least one vertex on T. Harary and Nash-Williams [9] proved the following result, showing that hamiltonicity of a line graph is equivalent to the existence of a dominating closed trail in its preimage.

Theorem B [9]. *Let H be a graph without isolated vertices. Then $L(H)$ is hamiltonian if and only if either H is isomorphic to $K_{1,r}$ (for some $r \geq 3$) or H contains a dominating closed trail.*

2 Main result

We begin with a brief overview of the history of consecutive improvements of minimum degree conditions for hamiltonicity in claw-free graphs. The first result in this direction was given by Dirac [2].

Theorem C [2]. *Let G be a graph of order $n \geq 3$ with minimum degree $\delta(G) \geq n/2$. Then G is hamiltonian.*

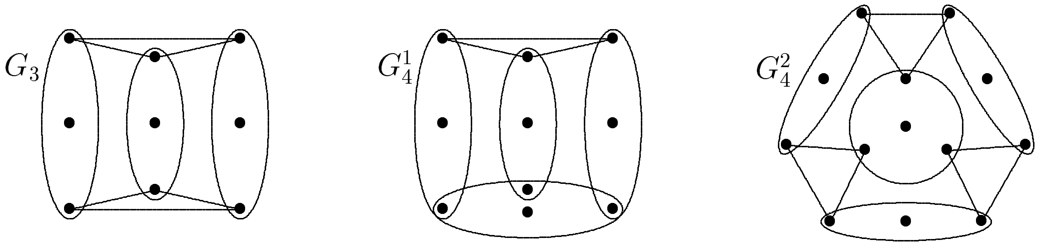
Although Dirac's condition is sharp in general, Matthews and Sumner [10] showed that it can be improved in the class of claw-free graphs.

Theorem D [10]. *Let G be a 2-connected claw-free graph of order n with minimum degree $\delta(G) \geq (n-2)/3$. Then G is hamiltonian.*

The graph G_3 in Figure 1 (where the elliptical parts represent cliques of appropriate order containing at least one simplicial vertex) shows that Theorem D is sharp. However, Hao Li [7] showed that this example is, in a sense, the only possible one. Let $\tilde{\mathcal{G}}_3$ be the class of all spanning subgraphs of the graph in Fig. 1.

Theorem E [7]. *Let G be a 2-connected claw-free graph of order n with minimum degree $\delta(G) \geq n/4$. Then either G is hamiltonian or $G \in \tilde{\mathcal{G}}_3$.*

The bound in Theorem E is sharp; however, Li, Lu, Tian and Wei [8] showed that another improvement was possible by enlarging the number of exceptions (for the class $\tilde{\mathcal{G}}_4$ see Figure 1).



$\tilde{\mathcal{G}}_3$ ($\tilde{\mathcal{G}}_4$) is the set of all spanning subgraphs of G_3 (G_4^1 and G_4^2)

Figure 1

Theorem F [8]. *Let G be a 2-connected claw-free graph of order n with minimum degree $\delta(G) \geq (n+5)/5$. Then either G is hamiltonian, or $G \in \tilde{\mathcal{G}}_3 \cup \tilde{\mathcal{G}}_4$.*

Theorem F is the strongest result in this direction that was achieved without using closure techniques.

Using the closure concept in claw-free graphs [11], Favaron, Flandrin, Li and Ryjáček [3] observed that there is a close relation between the minimum degree sum $\sigma_k(G)$ (or the minimum degree $\delta(G)$, respectively) of a closed claw-free graph G and its clique covering number. These connections are established in the following results [3].

Theorem G [3]. *Let $k \geq 2$ be an integer and let G be a claw-free graph of order n such that $\delta(G) > 3k-5$ and $\sigma_k(G) > n + k^2 - 2k$. Then $\theta(\mathrm{cl}(G)) \leq k-1$.*

Corollary H [3]. *Let $k \geq 2$ be an integer and let G be a claw-free graph of order $n \geq 2k^2 - 3k$ and minimum degree $\delta(G) > \frac{n}{k} + k - 2$. Then $\theta(\mathrm{cl}(G)) \leq k-1$.*

The bounds on $\sigma_k(G)$ ($\delta(G)$) in the previous results are sharp (this can be easily seen considering the cartesian product of cliques). However, these results can be improved under an additional assumption that G is not hamiltonian.

Theorem I [3]. *Let $k \geq 4$ be an integer and let G be a 2-connected claw-free graph with $|V(G)| = n$ such that $n \geq 3k^2 - 4k - 7$, $\delta(G) \geq 3k-4$ and*

$$\sigma_k(G) > n + k^2 - 4k + 7.$$

Then either $\theta(\mathrm{cl}(G)) \leq k-1$, or G is hamiltonian.

Corollary J [3]. *Let $k \geq 4$ be an integer and let G be a 2-connected claw-free graph with $|V(G)| = n$ such that $n \geq 3k^2 - 4k - 7$ and*

$$\delta(G) > \frac{n + k^2 - 4k + 7}{k}.$$

Then either $\theta(\mathrm{cl}(G)) \leq k-1$, or G is hamiltonian.

In [3], the classes of all 2-connected nonhamiltonian closed claw-free graphs with small clique covering number were listed for $\theta \leq 5$ using an exhaustive case-analysis. In this way, the following results were proved in [3] (for the class $\tilde{\mathcal{G}}_5$ see Figure 2).

Theorem K [3]. *Let G be a 2-connected claw-free graph with $n \geq 77$ vertices such that $\delta(G) \geq 14$ and*

$$\sigma_6(G) > n + 19.$$

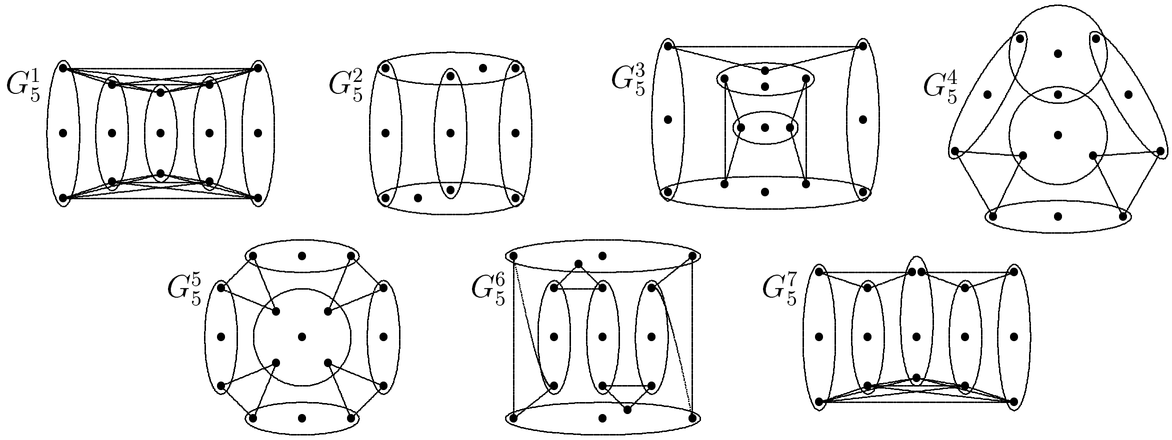
Then either G is hamiltonian or $G \in \tilde{\mathcal{G}}_3 \cup \tilde{\mathcal{G}}_4 \cup \tilde{\mathcal{G}}_5$.

Theorem K implies the following minimum degree result (which was also proved independently, using a different technique, by Kuipers and Veldman in [6]).

Corollary L [3], [6]. *Let G be a 2-connected claw-free graph of order $n \geq 78$ with*

$$\delta(G) \geq \frac{n+16}{6}.$$

Then either G is hamiltonian or $G \in \mathcal{G}_3 \cup \mathcal{G}_4 \cup \mathcal{G}_5$.



$\tilde{\mathcal{G}}_5$ is the set of spanning subgraphs of G_5^1, G_5^2, G_5^3, G_5^4, G_5^5, G_5^6 and G_5^7

Figure 2

Let $\mathcal{F}_i$ $(i = 3, \dots, 7)$ be the classes of graphs listed in the Appendix. For any $F_i^j \in \mathcal{F}_i$, let $\mathcal{G}_i^j$ be the set of all spanning subgraphs of all graphs obtained from the line graph $L(F_i^j)$ of F_i^j by adding an appropriate number of simplicial vertices to those cliques of $L(F_i^j)$ that correspond to the black vertices of F_i^j, and set $\mathcal{G}_i = \bigcup_{j=0}^{|\mathcal{F}_i|-1} \mathcal{G}_i^j$, $i = 3, \dots, 7$. Then it is easy to see that for $i = 3, 4, 5$, $\mathcal{G}_i = \{\mathrm{cl}(G) |\ G \in \tilde{\mathcal{G}}_i$ is 2-connected and claw-free$\}$, where $\tilde{\mathcal{G}}_i$ $(i = 3, 4, 5)$ are the classes of graphs from Figures 1 and 2.

In Sections 3 and 4 we present a method that was used for finding the classes $\mathcal{F}_6$ and $\mathcal{F}_7$ and establishing the fact that a 2-connected closed claw-free graph G with $\theta(G) \leq 7$ is nonhamiltonian if and only if $G \in \bigcup_{i=3}^{7} \mathcal{G}_i$. This result together with Theorem I and Corollary J yields the following theorem (which is the main result of this paper).

Theorem 1. *Let G be a 2-connected claw-free graph of order $n \geq 153$ such that $\delta(G) \geq 20$ and*

$$\sigma_8(G) > n + 39.$$

Then either G is hamiltonian or $G \in \bigcup_{i=3}^{7} \mathcal{G}_i$.

Theorem 1 immediately implies the following minimum degree result.

Corollary 2. *Let G be a 2-connected claw-free graph of order $n \geq 153$ with minimum degree*

$$\delta(G) \geq \frac{n + 39}{8}.$$

Then either G is hamiltonian or $G \in \bigcup_{i=3}^{7} \mathcal{G}_i$.

Proof of Theorem 1 and Corollary 2 follows immediately from Theorem I and Corollary J, respectively, and from the above mentioned properties of the classes $\mathcal{F}_i$. In the following sections we present the method which was used for the computer search for the classes of exceptions with $\theta = 6, 7$.

3 Preliminary observations

In this section we present basic definitions, notation and some auxiliary statements that will ensure the correctness and finiteness of the algorithm presented in Section 4.

We basically follow the terminology and notation introduced in [3]. Let $\mathcal{G}_\theta$ be the class of all 2-connected nonhamiltonian closed claw-free graphs with clique covering number θ. By Theorem A, every $G \in \mathcal{G}_\theta$ is the line graph of some (unique) triangle-free graph H. Let $D_1(H)$ be the set of all degree 1 vertices of H and put $H' = H - D_1(H)$. Set $\mathcal{H}_\theta = \{L^{-1}(G)|\ G \in \mathcal{G}_\theta\}$ and $\mathcal{H}'_\theta = \{H - D_1(H)|\ H \in \mathcal{H}_\theta\}$. Since every $G \in \mathcal{G}_\theta$ is 2-connected, every $H \in \mathcal{H}_\theta$ or $H' \in \mathcal{H}'_\theta$ is essentially 2-edge-connected or 2-edge-connected, respectively.

In every $G \in \mathcal{G}_\theta$ choose a fixed minimum clique covering $\mathcal{P}_G = \{B_1, \ldots, B_\theta\}$ of G such that each clique B_i is maximal. Since $\mathcal{P}_G$ is minimum, every B_i contains at least one *proper vertex*, i.e. a vertex belonging to no other clique of $\mathcal{P}_G$. The centers $B_1, \ldots, B_\theta$ of the stars of $H = L^{-1}(G)$ that correspond to the cliques of G will be called the *black vertices* of H. The other vertices of H are called *white*. The set of black (white) vertices of H is denoted by $B(H)$ ($W(H)$), respectively. Two vertices $b_1, b_2 \in B(H)$ are said to be *related* if they are either adjacent or they have a white common neighbor. Since $B(H)$ is a vertex covering of H (i.e., every edge of H has at least one vertex in $B(H)$), the set $W(H)$ is independent.

It is easy to see that for any $G \in \mathcal{G}_\theta$, any graph obtained from G by adding/removing simplicial vertices to/from cliques of $\mathcal{P}_G$ also belongs to $\mathcal{G}_\theta$ as long as (in the case of removal) at least one simplicial vertex in the clique remains (while the removal of the last simplicial vertex of a clique can turn G into a hamiltonian graph). Hence we can without loss of generality denote for any $H' \in \mathcal{H}'_\theta$ by $L(H)$ the line graph of H' in which one simplicial vertex is added to every clique corresponding to a black vertex of H'.

Let $G_1, G_2 \in \mathcal{G}_\theta$. We say that G_1 is an *ss-subgraph* of G_2, if G_1 is isomorphic to a spanning subgraph of a graph, which is obtained from G_2 by adding an appropriate number of simplicial vertices to some cliques of $\mathcal{P}_{G_2}$, and that G_1 is a *proper ss-subgraph* of G_2 if G_1 is an ss-subgraph of G_2 and G_1, G_2 are nonisomorphic. In the following we present a method for finding a subset $\mathcal{F}_\theta \subset \mathcal{H}'_\theta$ such that

(i) every $G \in \mathcal{G}_\theta$ is an ss-subgraph of $L(F)$ for some $F \in \mathcal{F}_\theta$,

(ii) for any $F_1, F_2 \in \mathcal{F}_\theta$, $L(F_1)$ is not an ss-subgraph of $L(F_2)$.

By the previous observations, the class $\mathcal{G}_\theta$ is fully characterized by $\mathcal{F}_\theta$.

If, for some $H \in \mathcal{H}_\theta$, the corresponding $H' \in \mathcal{H}'_\theta$ has a *black closed trail* (abbreviated BCT), i.e. a closed trail containing all black vertices of H', then clearly H has a DCT. Since, by Theorem B, no $H \in \mathcal{H}_\theta$ has a DCT, no $H' \in \mathcal{H}'_\theta$ has a BCT.

We say that a graph $H' \in \mathcal{H}'_\theta$ is *reducible* if there is a graph $H'_1 \in \mathcal{H}'_t$ for some $t \le \theta$ such that either

(i) H'_1 is obtained from H' by adding a relation (i.e., an edge or a white common neighbor) between two black vertices, or

(ii) $H'_1 = H'|_Y$ for some $Y \subset V(H')$ with $|Y| \ge 2$.

In the first case, we say that H' is *r-reducible*. In the second case, H' is said to be *ww-reducible* if $|Y \cap B(H')| = 0$, *bw-reducible* if $|Y \cap B(H')| = 1$, *bb-reducible* if $|Y \cap B(H')| \ge 2$.

The following statement shows that reducibility in $\mathcal{H}'_\theta$ is closely linked with ss-subgraphs in $\mathcal{G}_\theta$.

Theorem 3. *Let $G \in \mathcal{G}_\theta$, $H = L^{-1}(G) \in \mathcal{H}_\theta$ and $H' = H - D_1(H) \in \mathcal{H}'_\theta$. Then H' is reducible if and only if there is a graph $G_1 \in \mathcal{G}_t$ (for some $t \le \theta$) such that G is a proper ss-subgraph of G_1.*

Proof. 1. Suppose first that H' is reducible. If H'_1 is obtained by adding a relation between $b_i, b_j \in B(H')$, then $L(H'_1)$ is obtained from $L(H')$ either by adding a new vertex and joining it with all vertices of B_i and B_j (where B_i and B_j are the cliques corresponding to b_i and b_j) if an edge is added, or by adding a simplicial vertex to each of B_i, B_j and joining these vertices with an edge, if a relation with a new white vertex was added. In both cases, $L(H)$ is a proper ss-subgraph of $L(H'_1)$.

Let $H'_1 = H'|_Y$ for a set $Y \subset V(H')$ with $|Y| \ge 2$. Denote by Y_E the set of edges of H' that have at least one vertex in Y and by Y'_E the corresponding set of vertices of $L(H')$. Then $L(H'_1)$ can be equivalently obtained from $L(H')$ by adding all missing adges with both vertices in Y'_E (i.e., by making $\langle Y'_E \rangle$ a clique) and then removing an appropriate number of simplicial vertices. Thus, $L(H)$ is again a proper ss-subgraph of $L(H'_1)$.

2. Let now G be a proper ss-subgraph of some $G_1 \in \mathcal{G}_t$, $t \le \theta$. Then G is a spanning subgraph of a graph $G_1^S \in \mathcal{G}_t$, where G_1^S was obtained from G_1 by adding simplicial vertices to cliques of $\mathcal{P}_G$. Clearly also $G_1^S \in \mathcal{G}_t$; hence we can suppose $G_1^S = G_1$.

Let $uv \in E(G_1) \setminus E(G)$. Since $\mathcal{P}_G$ is a clique covering of G, there are cliques $B_u, B_v \in \mathcal{P}_G$ such that $u \in B_u \setminus B_v$ and $v \in B_v \setminus B_u$. Let b_u, b_v be the corresponding black vertices in H'.

First suppose that there is a vertex $z \in B_u \cap B_v$. Then, since $\{u, v, z\}$ induces a triangle and G_1 is closed, $\langle B_u \cup B_v \rangle_{G_1}$ is a clique. Hence H is bb-reducible (with $Y = \{b_u, b_v\}$). Thus, in the sequel we can suppose that $B_u \cap B_v = \emptyset$. We distinguish several cases.

<u>Case 1:</u> Both u and v are simplicial in G.

Then adding uv corresponds in H to adding a relation $b_1 w b_2$, where w is a (new) white vertex. Hence H is r-reducible.

Case 2: One of u, v is not simplicial in G.

By symmetry, suppose v is simplicial and u is not. Then $u \in K_u$ for some clique $K_u \subset V(G)$, $B_u \neq K_u \neq B_v$. Let $z_u \in V(H')$ be the vertex corresponding to K_u. If $K_u \cap B_v \neq \emptyset$, then, since G_1 is closed, $\langle K_u \cup B_v \rangle_{G_1}$ is a clique, implying H' is bw-reducible or bb-reducible with $Y = \{z_u, b_v\}$ (depending on whether z_u is white or black in H'). If $K_u \cap B_v = \emptyset$, then, since u cannot be a center of a claw in G_1, we have the following three possibilities.

- $\langle B_u \cup \{v\} \rangle_{G_1}$ is a clique. Then H' is r-reducible with adding the edge $b_u b_v$.
- $\langle K_u \cup \{v\} \rangle_{G_1}$ is a clique. Then similarly H' is r-reducible with adding the edge $z_u b_v$ (and hence, if z_u is white, the relation $b_u z_u b_v$).
- $\langle K_u \cup B_u \} \rangle_{G_1}$ is a clique. Then even the graph, obtained from G just by making $\langle K_u \cup B_u \} \rangle_{G_1}$ a clique (i.e. without adding uv) also belongs to $\mathcal{G}_t$, implying that H' is bb-reducible or bw-reducible with $Y = \{b_u, z_u\}$ (depending on whether z_u is black or white). The possibility of adding the edge uv then yields r-reducibility by Case 1.

Case 3: Neither u nor v is simplicial in G.

Then $v \in K_v$ for some clique K_v different from K_u, B_u, B_v. It is straightforward to check that, since neither u nor v can be a center of a claw, some two of the cliques K_u, K_v, B_u, B_v induce one clique, implying that H' is reducible. ∎

4 Algorithm

In this section we present the general idea of the algorithm used for generating all graphs from the classes $\mathcal{F}_6$ and $\mathcal{F}_7$. We do not give all technical details of the implementation. The interested reader can find this information in the thesis [5] which is (with the complete version of the source code of the program) available on www.

By Theorem 3, we have $\mathcal{F}_\theta = \{F \in \mathcal{H}'_\theta |\ F \text{ is ireducible}\}$. For any closed trail T in a graph $F \in \mathcal{F}_\theta$, denote by $\mathrm{bla}(T)$ the number of black vertices of T and by $\mathrm{blo}(T)$ the number of blocks of T. In every $F \in \mathcal{F}_\theta$ choose a closed trail T_F such that, among all closed trails in F,

(i) $\mathrm{bla}(T_F)$ is maximum,
(ii) subject to (i), $\mathrm{blo}(T_F)$ is minimum,
(iii) subject to (i) and (ii), T_F has minimum number of edges.

Such a T_F clearly exists and, since F has no BCT, $\mathrm{bla}(T_F) < \theta$. Hence every $F \in \mathcal{F}_\theta$ consists of the trail T_F with properties (i) – (iii), some black vertices outside T_F and some additional relations. This gives the following general idea of an algorithm for finding all graphs from the class $\mathcal{F}_\theta$.

Step 1. Generate all minimal closed trails T with $\mathrm{bla}(T) < \theta$.

Step 2. For each closed trail T from Step 1, generate all minimal 2-edge-connected graphs T^1, consisting of the trail T, additional $\theta - \mathrm{bla}(T)$ black vertices and connecting relations.

Step 3. Check each of the graphs T^1 from Step 2 for bb-reducibility. If T^1 is not bb-reducible, keep it (as T^2) for Step 4; otherwise generate all graphs T^2 obtained from T_1 by adding a minimal set of relations such that T^2 is not bb-reducible.

Step 4. a) For each of the graphs T^2 from Step 3, generate all graphs T^2_a, obtained from T^2 by adding all possible sets of relations that do not imply the existence of a closed trail T' with $\mathrm{bla}(T') > \mathrm{bla}(T)$ or with $\mathrm{bla}(T') = \mathrm{bla}(T)$ and $\mathrm{blo}(T') < \mathrm{blo}(T)$.

b) For each of the graphs T^2_a from Step 4a, create all possible graphs T^2_b by replacing by an edge all relations containing a white vertex of degree 2 for which the replacement does not yield a triangle.

Step 5. Check each of the graphs T^2_b from Step 4b for reducibility. $\mathcal{F}_\theta$ is the set of all ireducible graphs T^2_b.

In Step 1, for the considerations that follow, suppose without loss of generality that the generated minimal trails T are ordered in the order suggested by the preferences of the choice of T_F, i.e. in nonincreasing order of $\mathrm{bla}(T)$ and, for each value of $\mathrm{bla}(T)$, in nondecreasing order of $\mathrm{blo}(T)$.

In Step 2, the graphs T^1 are generated, for any fixed closed trail T from Step 1, by checking all possible relations between T and the vertices outside T. In order to reduce the number of cases to be considered, each of these graphs is checked for minimality (this can be supposed without loss of generality since the possibly missed relations are added later on in Step 4 anyway).

In Step 3, bb-reducibility of T^1 means that $L(T^1)$ is an ss-subgraph of a graph from $\mathcal{F}_t$ for some $t < \theta$, i.e. it is already known. However, this bb-reducibility can be due to some missing relations, and not considering this possibility could result in missing some cases.

In all steps from Step 1 to Step 4a, all realtions are supposed to contain a white vertex (this can be supposed without loss of generality since the white vertices which do not yield any new case are removed in Step 4b anyway).

In Step 4c, all further relations are added.

In all steps, the constructed graphs are checked for isomorphism with the previously generated (and stored) graphs.

In Steps 2 – 4, all constructed graphs are checked for nonexistence of a closed trail T' such that $\mathrm{bla}(T') > \mathrm{bla}(T)$ or $\mathrm{bla}(T') = \mathrm{bla}(T)$ and $\mathrm{blo}(T') < \mathrm{blo}(T)$ (where T is the closed trail that the graph under consideration was obtained from) since otherwise the subcase can be transformed to some of the previous ones. All constructed graphs are of course checked for being triangle-free.

It is clear that the algorithm, if it stops, yields all ireducible 2-edge-connected triangle-free graphs covered by a set of θ black vertices and with no BCT, i.e., by Proposition 3, the class $\mathcal{F}_\theta$.

Kuipers and Veldman [6] proved that, for each $\theta > 0$, the set $\mathcal{F}_\theta$ is finite (using a different, nonalgorithmic approach). Thus, to establish the finiteness of the algorithm, it suffices to show that the method used for generating closed trails in Step 1 always halts.

Let t be an integer and let T be a closed trail containing a covering set of $\mathrm{bla}(T) = t$ black vertices. Suppose that T is minimal (i.e., no proper subtrail of T contains all its black vertices). Let C_T be a cycle such that T is obtained from C_T by a series of identificatons of some of its vertices of the same color. For any $i \geq 1$ denote by $m_i(T)$ the number of black vertices of T that the trail T passes through i-times (i.e., the number of black vertices of degree $2i$). Then we have the following statement.

Theorem 4. *Let t be an integer and let T be a minimal closed trail with* $\mathrm{bla}(T) = t$. *Then*

(i) $m_1(T) + m_2(T) + \ldots = \mathrm{bla}(T)$,

(ii) if $m_j(T) \neq 0$ for some $j \geq 2$, then $m_1(T) \geq j$,

(iii) $m_j(T) = 0$ for $j \geq t$,

(iv) C_T has at most $\frac{1}{4}(t+1)^2$ black vertices.

Remark. Part *(iv)* of Theorem 4 establishes finiteness of Step 1 of the algorithm.

Proof. *(i)* Part *(i)* is straightforward.

(ii) Let $m_j(T) \neq 0$, let y be a vertex of T with degree $2j$ and suppose that y is obtained by identifying vertices $x_1, \ldots, x_j$ ($j \geq 2$) of C_T. If $m_1(T) < j$, then, in some of the segments of C_T between two consecutive x_i's (say, in $x_1 C_T x_2$), all interior black vertices are identified in T with some other vertices. But then the cycle $(x_1 = x_2) C_T x_1$ (obtained from C_T by identifying x_1 with x_2 and removing all interior vertices of the segment $x_1 C_T x_2$) yields a shorter trail T' with $\mathrm{bla}(T') = \mathrm{bla}(T)$, contradicting the minimality of T.

(iii) If $m_j(T) \neq 0$ for some $j \geq t$, then, by *(ii)*, $m_1(T) \geq j$, implying $t \geq m_1 + m_2 + \ldots + m_j \geq m_1 + m_j \geq j + 1 \geq t + 1$, a contradiction.

(iv) Let $m = \max\{j \geq 1 |\, m_j(T) \neq 0\}$. By *(iii)*, $m \leq t-1$, and by *(ii)*, $m_1(T) \geq m$. Hence at least m black vertices of T have degree 2, and the remaining at most $t - m$ black vertices of T have degree at most $2m$. This implies $\mathrm{bla}(C_T) \leq 1 \cdot m + m \cdot (t-m) = -m^2 + (t+1)m = \frac{(t+1)^2}{4} - (m - \frac{t+1}{2})^2 \leq \frac{(t+1)^2}{4}$. ■

Concluding remarks.

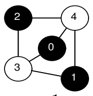
1. The algorithm was implemented in parallel on a cluster of 6 parallel workstations (6× Pentium Xeon 450 MHz, 6× 256 MB RAM, interconnection 1,6 Gb/s), running MPI (Message passing interface). A nonparallel version of the algorithm was also developed and implemented. The computing time of the parallel version was approx. 1 minute for $\theta = 6$ and 107 minutes for $\theta = 7$.

2. Generally speaking, it could be possible to obtain the exception classes even for larger values of θ. Nevertheless, the authors are convinced that a result presenting a degree condition for hamiltonicity in 2-connected claw-free graphs of type $\sigma_9(G) > n + 52$ (or, as a corollary, $\delta(G) > (n + 52)/9$) with a book of exceptions probably would not be very useful (although some of the exceptional graphs could be of interest on their own right). Thus, the authors believe that in the chase of improvements of degree conditions for hamiltonicity in 2-connected claw-free graphs there not much remains to be done.

Acknowledgement. The cluster used for the computation was built under a project MŠMT No. LB98246 "Lyra".

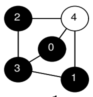
5 Appendix

5.1 Exception class $\mathcal{F}_3$

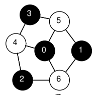


F_3^1

5.2 Exception class $\mathcal{F}_4$

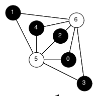


F_4^1

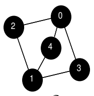


F_4^2

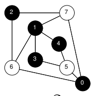
5.3 Exception class $\mathcal{F}_5$



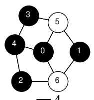
F_5^1



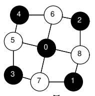
F_5^2



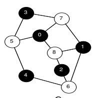
F_5^3



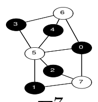
F_5^4



F_5^5

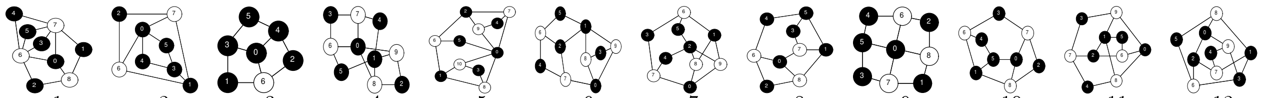


F_5^6

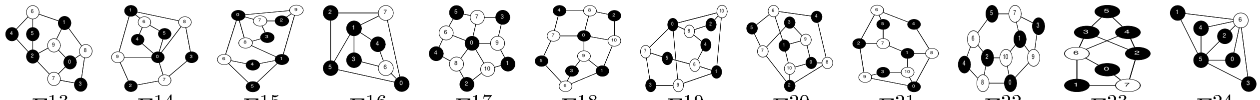


F_5^7

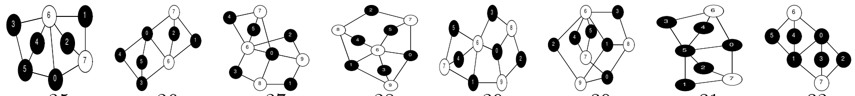
5.4 Exception class $\mathcal{F}_6$



F_6^1 F_6^2 F_6^3 F_6^4 F_6^5 F_6^6 F_6^7 F_6^8 F_6^9 F_6^{10} F_6^{11} F_6^{12}

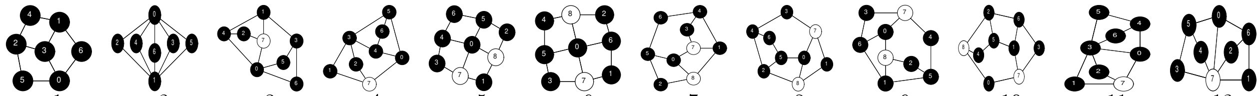


F_6^{13} F_6^{14} F_6^{15} F_6^{16} F_6^{17} F_6^{18} F_6^{19} F_6^{20} F_6^{21} F_6^{22} F_6^{23} F_6^{24}

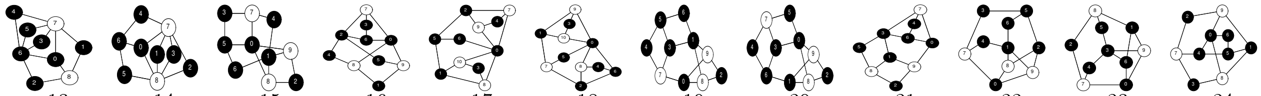


F_6^{25} F_6^{26} F_6^{27} F_6^{28} F_6^{29} F_6^{30} F_6^{31} F_6^{32}

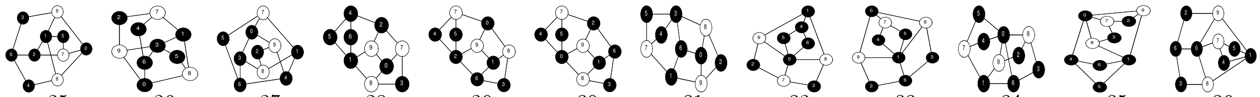
5.5 Exception class $\mathcal{F}_7$



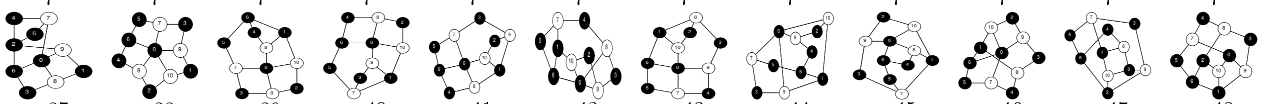
F_7^1 F_7^2 F_7^3 F_7^4 F_7^5 F_7^6 F_7^7 F_7^8 F_7^9 F_7^{10} F_7^{11} F_7^{12}



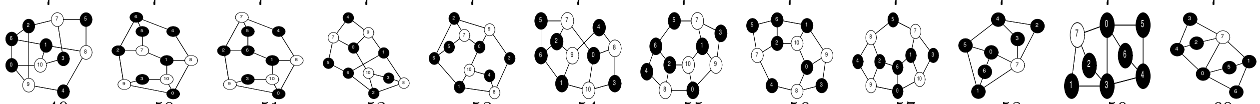
F_7^{13} F_7^{14} F_7^{15} F_7^{16} F_7^{17} F_7^{18} F_7^{19} F_7^{20} F_7^{21} F_7^{22} F_7^{23} F_7^{24}



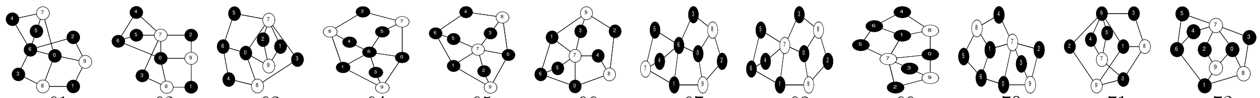
F_7^{25} F_7^{26} F_7^{27} F_7^{28} F_7^{29} F_7^{30} F_7^{31} F_7^{32} F_7^{33} F_7^{34} F_7^{35} F_7^{36}



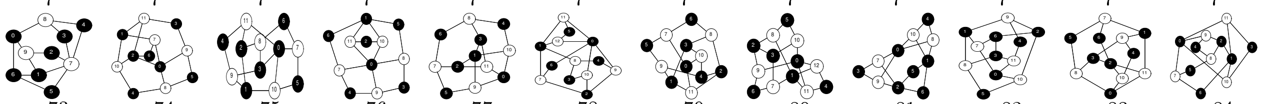
F_7^{37} F_7^{38} F_7^{39} F_7^{40} F_7^{41} F_7^{42} F_7^{43} F_7^{44} F_7^{45} F_7^{46} F_7^{47} F_7^{48}



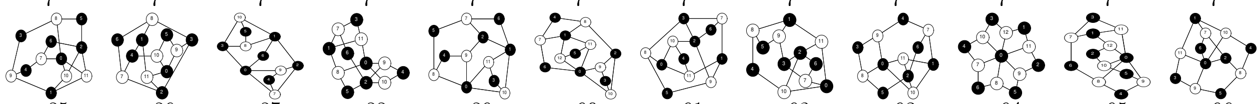
F_7^{49} F_7^{50} F_7^{51} F_7^{52} F_7^{53} F_7^{54} F_7^{55} F_7^{56} F_7^{57} F_7^{58} F_7^{59} F_7^{60}



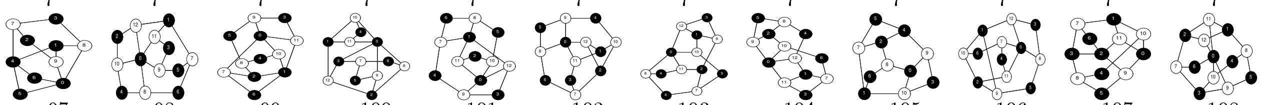
F_7^{61} F_7^{62} F_7^{63} F_7^{64} F_7^{65} F_7^{66} F_7^{67} F_7^{68} F_7^{69} F_7^{70} F_7^{71} F_7^{72}



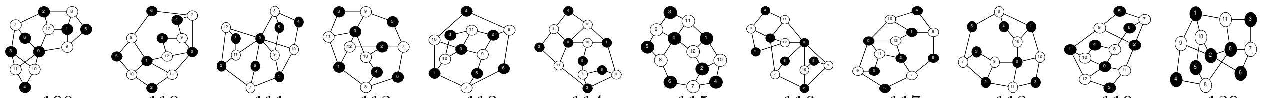
F_7^{73} F_7^{74} F_7^{75} F_7^{76} F_7^{77} F_7^{78} F_7^{79} F_7^{80} F_7^{81} F_7^{82} F_7^{83} F_7^{84}



F_7^{85} F_7^{86} F_7^{87} F_7^{88} F_7^{89} F_7^{90} F_7^{91} F_7^{92} F_7^{93} F_7^{94} F_7^{95} F_7^{96}



F_7^{97} F_7^{98} F_7^{99} F_7^{100} F_7^{101} F_7^{102} F_7^{103} F_7^{104} F_7^{105} F_7^{106} F_7^{107} F_7^{108}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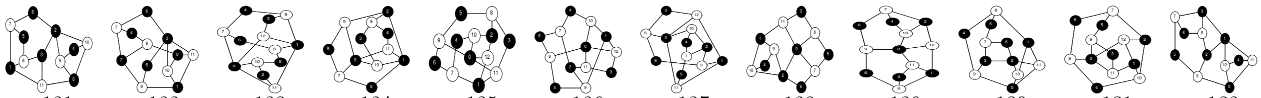
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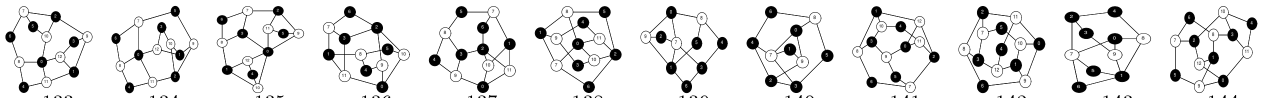
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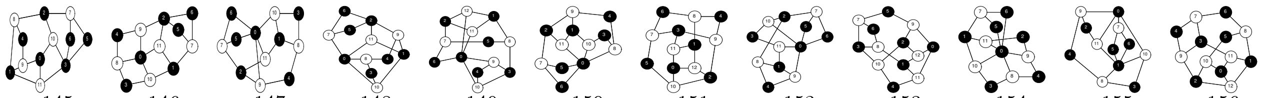
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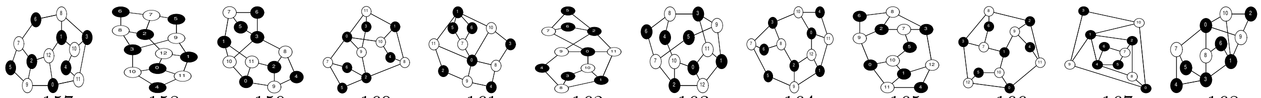
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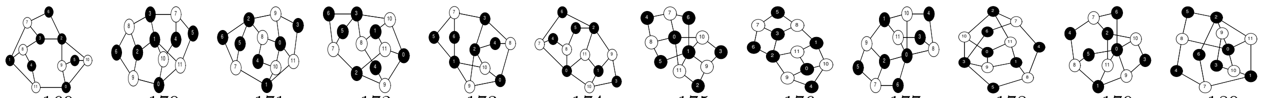
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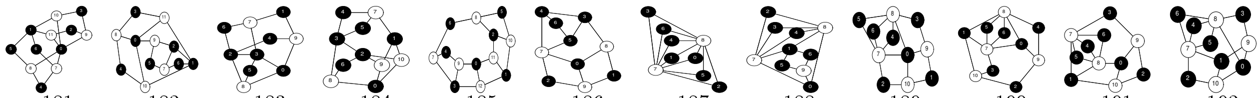
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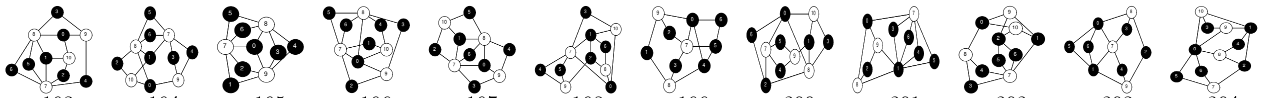
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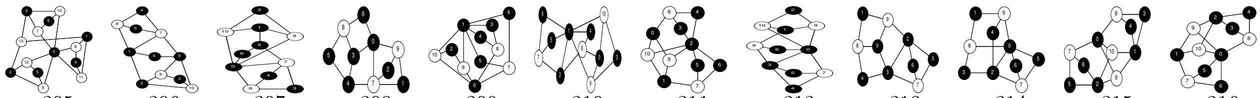
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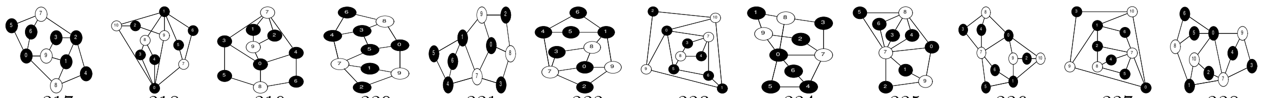
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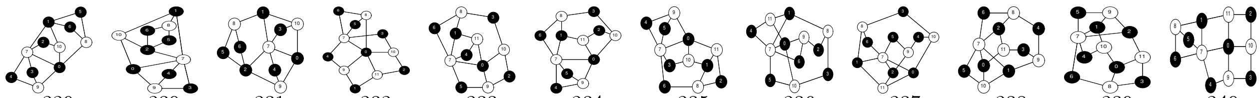
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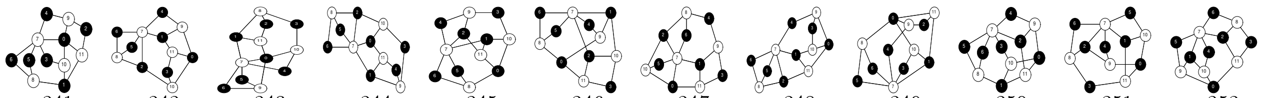
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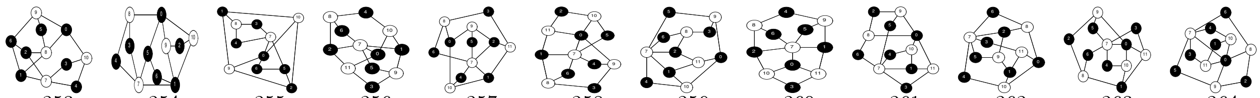
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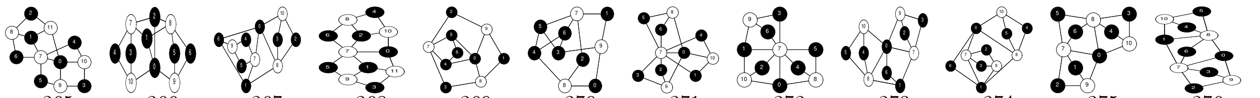
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