Forbidden subgraphs implying the MIN-algorithm gives a maximum independent set

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December 30, 2001

Abstract

The well-known greedy algorithm MIN for finding a maximal independent set in a graph G is based on recursively removing the closed neighborhood of a vertex which has (in the currently existing graph) minimum degree. We give a forbidden induced subgraph condition under which algorithm MIN always results in finding a maximum independent set of G, and hence yields the exact value of the independence number of G in polynomial time.

*Research supported by grant GA ČR No. 201/97/0407

1 Introduction

Throughout the paper, we consider only finite undirected graphs $G = (V(G), E(G))$ without loops and multiple edges. By $N_G(x)$ we denote the *neighborhood* of a vertex $x \in V(G)$, i.e., the set of all neighbors of x. We further denote by $N_G[x] = N_G(x) \cup \{x\}$ the *closed neighborhood* of x in G, by $d_G(x) = |N_G(x)|$ the *degree* of x in G and by $\delta(G) = \min\{d_G(x) | \, x \in V(G)\}$ the *minimum degree* of G. For a set $M \subset V(G)$ we denote by $\langle M \rangle_G$ the induced subgraph of G on M and we set $G - M = \langle V(G) \setminus M \rangle_G$. By $\alpha(G)$ we denote the *independence number* of G, i.e., the size of a maximum (i.e. largest) independent set in G. If $F_1, \ldots, F_k$ are graphs, then we say that G is $\{F_1, \ldots, F_k\}$*-free* if G does not contain a copy of any of the graphs $F_1, \ldots, F_k$ as an induced subgraph. For other terminology and notation not defined here we refer to [1].

The well-known greedy algorithm MIN for finding a maximal independent set in a graph G [4] can be stated as follows.

Algorithm MIN (Minimum degree).

1. *$H_1 := G$; $i := 1$; $S_{MIN} := \emptyset$.*
2. *Choose a vertex $v_i \in V(H_i)$ such that $d_{H_i}(v_i) = \delta(H_i)$ and set $S_{MIN} := S_{MIN} \cup \{v_i\}$; $H_{i+1} := H_i - N_{H_i}[v_i]$.*
3. *If $V(H_{i+1}) \neq \emptyset$ then $i := i + 1$ and go to 2.*
4. *STOP.*

Obviously, the set S_{MIN}, generated by Algorithm MIN, is a maximal (but not necessarily maximum) independent set in G, and hence $\alpha(G) \geq |S_{MIN}|$.

Mahadev and Reed [3] considered the following (also greedy) algorithm for finding a maximal independent set in G, based on an ordering of the vertices of G according to their degrees in G. This algorithm can be equivalently formulated as follows.

Algorithm VO (Vertex order).

1. *Order the vertices of G into a sequence $v_1, \ldots, v_n$ such that $d_G(v_j) \leq d_G(v_k)$ for any j, k, $1 \leq j < k \leq n$.*
2. *$G_1 := G$; $i := 1$; $S_{VO} := \emptyset$.*
3. *For $i := 1$ to n do:*
   If $N_G(v_i) \cap S_{VO} = \emptyset$, then $S_{VO} := S_{VO} \cup \{v_i\}$.
4. *STOP.*

It is clear that the set S_{VO}, generated by Algorithm VO, is a maximal independent set in G, and hence also $\alpha(G) \geq |S_{VO}|$.

Note that both Algorithm MIN and Algorithm VO have polynomial time complexity whereas the determination of $\alpha(G)$ is difficult since the corresponding decision problem INDEPENDENT SET is a well-known NP-complete problem [2].

Denote by $k_{MIN}(G)$ and $k_{VO}(G)$ the smallest cardinality of an independent set of G that Algorithm MIN and Algorithm VO can create, respectively. Let $F_1, \ldots, F_6$ be the graphs in Fig. 1 and let $\mathcal{F}_A = \{F_1, F_2, F_3, F_4, F_5, F_6\}$.

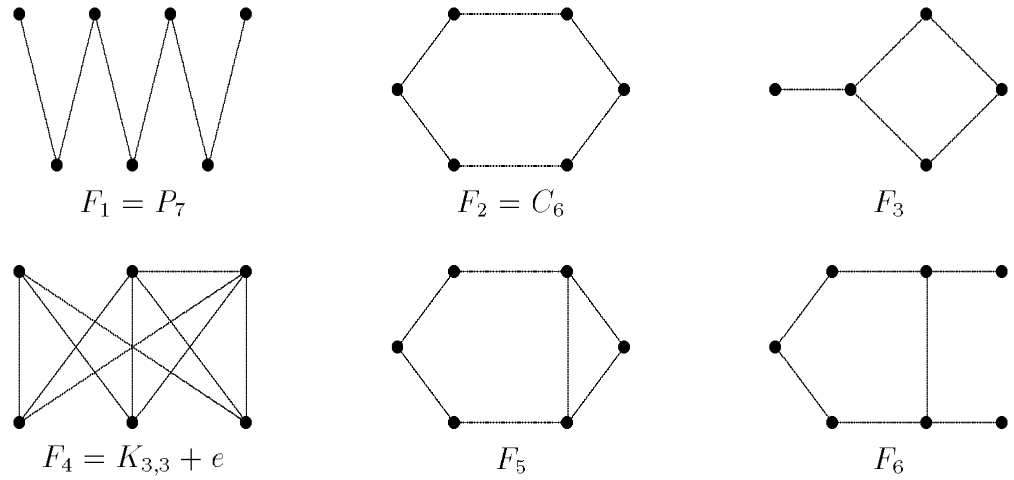

Figure 1

The following theorem, which forms the essential part of the main result of [3], shows that in the class of $\mathcal{F}_A$-free graphs Algorithm VO always yields a maximum independent set.

Theorem A [3]. *Let G be an $\mathcal{F}_A$-free graph. Then*

$$k_{VO}(G) = \alpha(G).$$

2 Main result

Let $F_7, \ldots, F_{13}$ be the graphs shown in Fig. 2 and let $\mathcal{F}_1 = \{F_1, F_3, F_5, F_6, F_7, F_8, F_9, F_{10}, F_{11}, F_{12}, F_{13}\}$.

Since F_2 is an induced subgraph of F_7 and F_4 is an induced subgraph of each of the graphs $F_8, \ldots, F_{13}$, the class of $\mathcal{F}_A$-free graphs is a proper subclass of the class of $\mathcal{F}_1$-free graphs. Thus, the following theorem, which is the main result of this paper, extends Theorem A in the sense that even for $\mathcal{F}_1$-free graphs the independence number can be calculated in polynomial time.

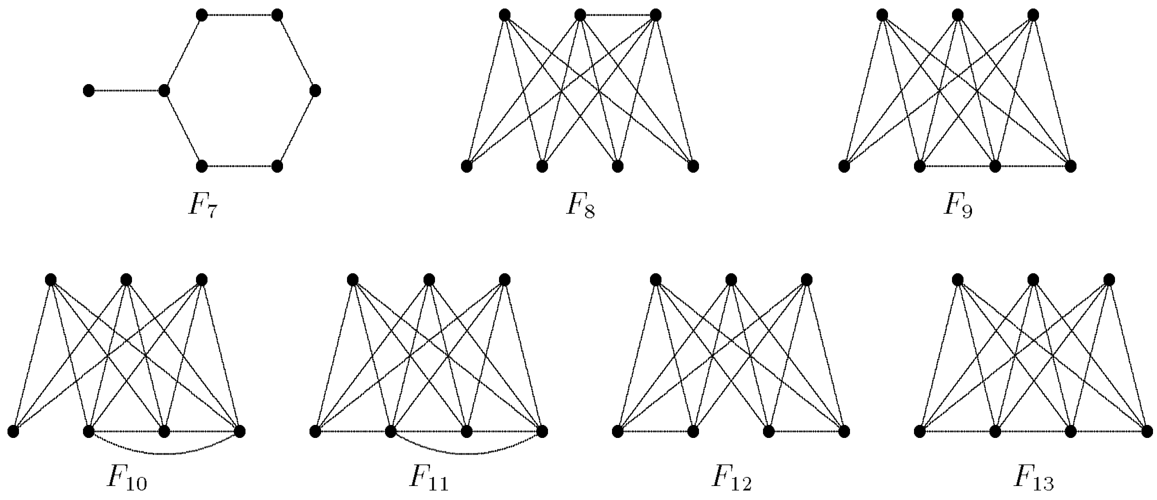

Figure 2

Theorem 1. *Let G be an $\mathcal{F}_1$-free graph of order $n \geq 7$. Then*

$$k_{MIN}(G) = \alpha(G).$$

Equivalently, Theorem 1 gives a collection of forbidden induced subgraphs which imply that Algorithm MIN always yields a maximum independent set. The proof of Theorem 1 is postponed to Section 3.

As already noted, $\mathcal{F}_A$-free $\Rightarrow$ $\mathcal{F}_1$-free. However, the price for a more general result is paid here in larger number of forbidden subgraphs. The following corollary of Theorem 1 avoids this drawback and still extends Theorem A.

Let $\mathcal{F}_2 = \{F_1, F_3, F_4, F_5, F_6, F_7\}$. Note that, since F_7 contains an induced F_2 and each of the graphs $F_8, \ldots, F_{13}$ contains an induced F_4, we have $\mathcal{F}_A$-free $\Rightarrow$ $\mathcal{F}_2$-free $\Rightarrow$ $\mathcal{F}_1$-free.

Corollary 2. *Let G be an $\mathcal{F}_2$-free graph of order $n \geq 7$. Then*

$$k_{MIN}(G) = \alpha(G).$$

The following statement shows that Corollary 2 (and hence also Theorem 1) is considerably stronger than Theorem A. More specifically, it says that under the assumptions of Corollary 2 the difference between the output of Algorithm MIN and that of Algorithm VO can be arbitrarily large.

Theorem 3. *For every integer k there is an $\mathcal{F}_2$-free graph G such that*

$$k_{MIN}(G) - k_{VO}(G) \geq k.$$

Proof. Let $\mathcal{G}$ be the class of graphs defined recursively as follows:

(i) $F_2 \in \mathcal{G}$,

(ii) for any $G_1, G_2 \in \mathcal{G}$ let also $(G_1 + G_2) \vee K_1 \in \mathcal{G}$ and $(G_1 + G_2) \vee \overline{K_2} \in \mathcal{G}$.

(Following [1], we denote by "+" the disjoint union and by "$\vee$" the join of two graphs, respectively).

We show that every graph $G \in \mathcal{G}$ is $\mathcal{F}_2$-free. We first have the following observation, the proof of which is obvious.

Claim. *Let $F \in \mathcal{F}_2$ with $|V(F)| = r$. Then $d_F(x) \leq r - 2$ for every $x \in V(F)$ and* $\min\{d_F(x), d_F(y)\} \leq r - 3$ *for any pair of independent vertices $x, y \in V(F)$.* □

Since $F_2 \notin \mathcal{F}_2$, the graph F_2 is $\mathcal{F}_2$-free. Suppose now that G_1, G_2 are $\mathcal{F}_2$-free. If $(G_1 + G_2) \vee K_1$ or $(G_1 + G_2) \vee \overline{K_2}$ contains an induced $F \in \mathcal{F}_2$, then, since F is connected, $V(F)$ contains at least one vertex outside $V(G_1) \cup V(G_2)$, but then we have a contradiction with the claim. Hence every graph in $\mathcal{G}$ is $\mathcal{F}_2$-free.

If we now set $G'_1 = F_2$ and $G'_{i+1} = (G'_i + G'_i) \vee K_1$ for $i \geq 1$, then $G'_i \in \mathcal{G}$ for any $i \geq 1$ and it is apparent that $k_{MIN}(G'_i) = \alpha(G'_i) = 3 \cdot 2^{i-1}$, but $k_{VO}(G'_i) = 2^i$. ■

Remark. By Theorem 1, in the class of $\mathcal{F}_1$-free graphs Algorithm MIN is always at least as good as Algorithm VO and by Theorem 3 the difference can be arbitrarily large. The following construction shows that without the assumption of $\mathcal{F}_1$-freeness Algorithm VO can be better than Algorithm MIN (i.e., for all graphs, the two algorithms are incomparable).

Let $p \geq 3$ be an arbitrary integer, let $G^1 \simeq G^2 \simeq K_p$, $G^3 \simeq K_1$ and $G^4 \simeq \overline{K_p}$ be vertex-disjoint, let G'_p be the graph obtained by joining by an edge all pairs of vertices x, y for $x \in V(G^i)$, $y \in V(G^{i+1})$ (mod 4), and let G_p be the graph obtained by adding one new vertex to G'_p and joining it to all vertices of G^2. Then clearly $k_{MIN}(G_p) = 3$, while $k_{VO}(G_p) = p + 1$.

Since Algorithm MIN is (clearly) polynomial, we further have the following consequence of Theorem 1.

Corollary 4. *In the class of $\mathcal{F}_1$-free graphs the independence number can be computed in polynomial time.*

■

Note that it is obvious that $\mathcal{F}_1$-free graphs are recognizable in polynomial time.

3 Proof of Theorem 1

We basically follow the general idea of the proof of Theorem A in [3], with replacing Algorithm VO by Algorithm MIN and the set $\mathcal{F}_A$ by the set $\mathcal{F}_1$. For the sake of clarity, whenever we list vertices of some induced subgraph F, we always order the vertices of the list such that their degrees (in F) form a nonincreasing sequence (with the exception of $F_1 \simeq P_7$, where the ordering follows the path).

Let G be a (without loss of generality) connected graph satisfying the assumptions of Theorem 1 and suppose that Algorithm MIN creates a maximal independent set S in G such that $|S| = m < \alpha(G)$, i.e., such that S is not maximum. Let the notation of v_i, H_i be chosen in accordance with the description of Algorithm MIN in Section 1, i.e., such that $S = \{v_1, \ldots, v_m\}$, $H_1 = G$, $d_{H_i}(v_i) = \delta(H_i)$ and $H_{i+1} = H_i - N[v_i]$, and set $S_j = S \cap V(H_j) = \{v_j, \ldots, v_m\}$, $j = 1, \ldots, m$. Choose a maximum independent set $T = \{t_1, \ldots, t_\alpha\}$ in G such that $|S \cap T|$ is maximum, and set $T_j = T \cap V(H_j)$, $j = 1, \ldots, m$. Since both S and T are independent, $\langle S \cup T \rangle_G$ is bipartite with all its isolated vertices in $S \cap T$. Let R be a component of $\langle S \cup T \rangle_G$ with $|R \cap S| < |R \cap T|$ (such an R always exists since $|S| < |T|$) and set $k = \min\{i \in \{1, \ldots, m\} | \; v_i \in R \cap S\}$ (with a slight abuse of notation, we will use R for both the component and its vertex set).

We have the following observations.

Claim 1. *S_j is a dominating set in H_j, $j = 1, \ldots, m$.*

Proof. If $x \in V(H_j) \setminus S_j$, then $N_G(x) \cap \{v_1, \ldots, v_{j-1}\} = \emptyset$, since otherwise $x \notin V(H_j)$ by the definition of H_j. Since S is a dominating set in G, necessarily $N_G(x) \cap S_j \neq \emptyset$, implying $N_{H_j}(x) \cap S_j \neq \emptyset$. □

Claim 2. *$d_{H_j}(x) \geq d_{H_j}(v_j)$ for every $x \in V(H_j)$ and for every $j = 1, \ldots, m$.*

Proof follows immediately from the definition of Algorithm MIN.

Claim 3. *$R \subset V(H_k)$.*

Proof. Obviously $R \cap S \subset V(H_k)$. If $y \in (R \cap T) \setminus V(H_k)$, then $y \in N_{H_j}(v_j)$ for some $j < k$ and hence $v_j \in R \cap S$, contradicting the choice of k. Hence also $R \cap T \subset V(H_k)$. □

The following simple observation will be often used implicitly throughout the proof.

Claim 4. *If F is a subgraph of H_j for some $j \in \{1, \ldots, m\}$, then F is induced in H_j if and only if F is induced in G.* □

In the sequel, we will use the following notation: $|R \cap S| = p$, $|R \cap T| = q$, $R \cap S = \{v_{i_1}, \ldots, v_{i_p}\}$, $R \cap T = \{t_1, \ldots, t_q\}$, and we suppose the notation of the vertices in $R \cap S$ is chosen such that $i_1 = k$ and $i_{j_1} < i_{j_2}$ for $j_1 < j_2$.

Case 1: R contains a cycle.

If R contains an induced cycle of length $\ell \geq 8$, then R contains also an induced F_1, a contradiction.

Suppose that R contains an induced cycle C of length 6, and let $C = s_1t_1s_2t_2s_3t_3s_1$, where $s_i \in R \cap S$, $t_i \in R \cap T$, $i = 1,2,3$. Since $|R \cap S| < |R \cap T|$, there is a $t_4 \in R \cap T$, adjacent to (say) s_1. If $s_2t_4 \in E(G)$, then (since C is induced and T is independent), $\langle\{s_2, t_4, s_1, t_1, t_2\}\rangle_G \simeq F_3$, a contradiction. Hence $s_2t_4 \notin E(G)$, and similarly $s_3t_4 \notin E(G)$. But then $\langle\{s_1, t_1, s_2, t_2, s_3, t_3, t_4\}\rangle_G \simeq F_7$, a contradiction. Hence every induced cycle in R has length exactly 4. Since R is bipartite and F_3-free, it follows easily (by induction, starting with a C_4) that R is a complete bipartite graph with $2 \leq |R \cap S| < |R \cap T|$.

Consider the vertex $v_k \in R \cap S$. We have $d_R(v_k) > d_R(y)$ for every $y \in R \cap T$ (since $|R \cap S| < |R \cap T|$), but, on the other hand, by the choice of v_k and by Claim 2, $d_{H_k}(v_k) \leq d_{H_k}(y)$ for every $y \in R \cap T$. It follows that there are vertices $z \in V(H_k) \setminus R$ and $y \in R \cap T$ such that $zy \in E(G)$, but $zv_k \notin E(G)$.

Claim 5. *Let $z \in V(H_k) \setminus R$ be such that $zv_k \notin E(G)$ and $N_{R\cap T}(z) \neq \emptyset$. Then $N_{R\cap S}(z) \neq \emptyset$ and $N_{R\cap T}(z) = \{t_1, \ldots, t_q\}$.*

Proof. Let (without loss of generality) $zt_1 \in E(G)$. Suppose first that $N_{R\cap S}(z) = \emptyset$. Then $N_R(z) = R \cap T$, since otherwise $\langle R \cup \{z\}\rangle_{H_k}$ contains an induced F_3. Since S is dominating, $zs \in E(G)$ for some $s \in S \setminus R$. Then $N_{R\cap S}(s) = \emptyset$ (since S is independent) and $N_{R\cap T}(s) = \emptyset$ (otherwise $s \in R$), implying $N_R(s) = \emptyset$ and $\langle\{z, t_1, v_k, t_2, s\}\rangle_G \simeq F_3$. Hence $N_{R\cap S}(z) \neq \emptyset$.

Let (without loss of generality) $zv_{i_2} \in E(G)$. Recall that $i_1 = k$, i.e., $v_{i_1} = v_k$. If $zt_a, zt_b \notin E(G)$ for some $a, b \in \{2, \ldots, q\}$, then $\langle\{v_{i_2}, t_a, v_{i_1}, t_b, z\}\rangle_G \simeq F_3$, and if $zt_a \notin E(G)$ and $zt_b \in E(G)$ for some $a, b \in \{2, \ldots, q\}$, then $\langle\{v_{i_1}, t_1, z, t_b, t_a\}\rangle_G \simeq F_3$. Hence $zt_i \in E(G)$ for every $i = 1, \ldots, q$. □

Now, by Claim 5, $q \geq 4$ implies $\langle\{z, v_{i_2}, v_{i_1}, t_1, t_2, t_3, t_4\}\rangle_G \simeq F_8$. Hence $q = 3$ and, consequently, $p = 2$.

Denote $H = \langle\{z, v_{i_2}, v_{i_1}, t_1, t_2, t_3\}\rangle_G$ (note that $H \simeq F_4$). Since $|V(G)| \geq 7$, there is a vertex $y \in V(G) \setminus V(H)$ with $N_H(y) \neq \emptyset$.

Suppose first that $yv_{i_1} \in E(G)$. If $y \in V(H_k)$ and $yt_i \in E(G)$ for $i = 1,2,3$, then $\langle\{y, z, v_{i_1}, v_{i_2}, t_1, t_2, t_3\}\rangle_G$ is isomorphic to one of the graphs F_{11}, F_{12} or F_{13}, depending on the existence of the edges yv_{i_2}, yz. If $y \in V(H_k)$ and (say) $yt_1 \notin E(G)$, then, by the choice of v_k (as a vertex of minimum degree in H_k) and by Claim 2, there is a $z' \in V(H_k) \setminus V(H)$ such that $z'v_{i_1} \notin E(G)$ but $z't_1 \in E(G)$. By Claim 5, $\{v_{i_2}, t_1, t_2, t_3\} \subset N_H(z')$, i.e., $\langle V(H) \setminus \{z\} \cup \{z'\}\rangle_G \simeq F_4$. Then $\langle V(H) \cup \{z'\}\rangle_G$ induces F_{10} or F_9, depending on whether $zz' \in E(G)$ or not.

If $y \notin V(H_k)$, then $yv_{i_0} \in E(G)$ for some i_0, $1 \leq i_0 < k$. Note that $N_H(v_{i_0}) = \emptyset$ (since $i_0 < k$). Then either $N_H(y) = V(H)$, implying $\langle V(H) \cup \{y\}\rangle_G \simeq F_{11}$, or y is nonadjacent to some vertex of H, and then it is easy to see that $\langle V(H) \cup \{y, v_{i_0}\}\rangle_G$ contains an induced F_3 for any possible structure of $N_H(y)$. This contradiction proves that $yv_{i_1} \notin E(G)$.

If $N_{R\cap T}(y) = \emptyset$, then $yz \in E(G)$ or $yv_{i_2} \in E(G)$, but in both cases we have an induced F_3. Hence $N_{R\cap T}(y) \neq \emptyset$. By Claim 5, $yv_{i_2} \in E(G)$ and $yt_i \in E(G)$ for $i = 1, 2, 3$. Then again $\langle V(H) \cup \{y\}\rangle_G$ induces an F_{10} or F_9, depending on whether $yz \in E(G)$ or not. This contradiction completes the proof in Case 1.

Case 2: R is a tree.

Claim 6. *All leaves of R are in T.*

Proof. If $s \in S$ is a leaf of R and $t \in T$ is the (only) neighbor of s in R, then $T \setminus \{t\} \cup \{s\}$ is also a maximum independent set, contradicting the maximality of $|S \cap T|$. □

Claim 6 immediately implies that every longest path in R has an odd number of vertices. Since G is F_1-free, a longest path in R can be only a P_3 or a P_5.

Subcase 2.1: R contains a P_3 both endvertices of which are leaves of R.

By Claim 6, let $t_a v_{i_\ell} t_b$ (where $1 \leq \ell \leq p$ and $1 \leq a, b \leq q$) be the vertices of the P_3. First observe that $t_a, t_b \in V(H_{i_\ell})$ (since otherwise e.g. $t_a \notin V(H_{i_\ell})$ would imply $t_a v_c \in E(G)$ for some c, $1 \leq c < i_\ell$, but then for $1 \leq c < k$ the vertex t_a would not be in H_k, and for $k \leq c < i_\ell$ the vertex t_a would not be a leaf of R). By Claim 2, there are vertices $x_a, x_b \in V(H_{i_\ell}) \setminus R$ such that $x_a t_a \in E(G)$ and $x_b t_b \in E(G)$, but $x_a v_{i_\ell}, x_b v_{i_\ell} \notin E(G)$. By Claim 1, each of x_a, x_b has a neighbor (say, $v_{a'}$ and $v_{b'}$) in S_{i_ℓ}. Note that $v_{a'}, v_{b'}$ are nonadjacent to t_a and t_b (otherwise t_a, t_b are not leaves). Now we have $x_a \neq x_b$ (otherwise $\langle\{x_a, t_a, v_{i_\ell}, t_b, v_{a'}\}\rangle_G \simeq F_3$), $x_a t_b \notin E(G)$ and $x_b t_a \notin E(G)$ (otherwise $\langle\{x_a, t_b, v_{i_\ell}, t_a, v_{a'}\}\rangle_G \simeq F_3$ or $\langle\{x_b, t_a, v_{i_\ell}, t_b, v_{b'}\}\rangle_G \simeq F_3$) and, finally, $x_a x_b \notin E(G)$ and $v_{a'} = v_{b'}$ (otherwise $\langle\{x_a, x_b, t_a, t_b, v_{i_\ell}, v_{a'}, v_{b'}\}\rangle_G$ induces F_1, F_6 or F_5).

Since the vertex $v_{a'}$ $(= v_{b'})$ is in S_{i_ℓ} (but not necessarily in R), we have $v_{a'} = v_{\ell'}$ for some ℓ', $i_\ell < \ell' \leq m$. Suppose that, among all common neighbors of x_a, x_b in S_{i_ℓ}, $v_{a'}$ is chosen such that ℓ' is minimum. Then $x_a, x_b \in H_{\ell'}$, but $t_a, t_b \notin H_{\ell'}$. By Claim 2 (for $j = \ell'$), there are $z_a, z_b \in V(H_{\ell'})$ such that $z_a x_a \in E(G)$ and $z_b x_b \in E(G)$, but $z_a, z_b \notin N_G(v_{\ell'})$ (and also $z_a, z_b \notin N_G(v_{i_\ell})$).

Suppose first that $z_a = z_b$. Since $\langle\{x_a, z_a, x_b, v_{\ell'}, t_a\}\rangle_G \not\simeq F_3$, we have $t_a z_a \in E(G)$. Symmetrically, $\langle\{x_b, z_a, x_a, v_{\ell'}, t_b\}\rangle_G \not\simeq F_3$ implies $t_b z_a \in E(G)$. Then $z_a \notin S_{\ell'}$ (otherwise t_a, t_b are not leaves). By Claim 1, z_a has a neighbor s in S, but then $\langle\{z_a, t_a, v_{i_\ell}, t_b, s\}\rangle_G \simeq F_3$. Hence $z_a \neq z_b$ (implying $z_a x_b \notin E(G)$ and $z_b x_a \notin E(G)$).

We show that $z_a t_a \notin E(G)$. Let $z_a t_a \in E(G)$. If $z_a t_b \in E(G)$, then $\langle\{t_b, v_{i_\ell}, t_a, z_a, x_b\}\rangle_G \simeq F_3$; hence $z_a t_b \notin E(G)$. Clearly $z_a \notin S$ (otherwise t_a is not a leaf) and hence, by Claim 1, z_a has a neighbor s_a in S. Obviously, s_a is not adjacent to any of $v_{i_\ell}, v_{\ell'}, t_a, t_b$. If $s_a x_b \notin E(G)$, then $\langle\{s_a, z_a, t_a, v_{i_\ell}, t_b, x_b, v_{\ell'}\}\rangle_G \simeq F_1$; hence $s_a x_b \in E(G)$, but then for $s_a x_a \in E(G)$ we have $\langle\{x_b, v_{\ell'}, x_a, s_a, t_b\}\rangle_G \simeq F_3$, and for $s_a x_a \notin E(G)$ we have $\langle\{x_a, z_a, s_a, x_b, v_{\ell'}, t_a\}\rangle_G \simeq F_5$. Hence $z_a t_a \notin E(G)$.

Since $\langle\{x_a, t_a, v_{i_\ell}, t_b, x_b, v_{\ell'}, z_a\}\rangle_G \not\simeq F_7$, we obtain $z_a t_b \in E(G)$. Symmetrically, $z_b t_b \notin E(G)$ and $z_b t_a \in E(G)$. This also implies that $z_a, z_b \notin S$ (otherwise t_a or t_b is not a leaf). By Claim 1, there are vertices $s_a, s_b \in S_{\ell'}$ such that $z_a s_a \in E(G)$ and $z_b s_b \in E(G)$ (possibly $s_a = s_b$). Obviously, s_a and s_b are not adjacent to any of $t_a, t_b, v_{i_\ell}, v_{\ell'}$. If $s_a x_a \in E(G)$, then for $s_a = s_b$ and $x_b s_b \in E(G)$ we have $\langle\{x_a, v_{\ell'}, x_b, s_a, t_a\}\rangle_G \simeq F_3$, otherwise $\langle\{x_a, t_a, v_{i_\ell}, t_b, x_b, v_{\ell'}, s_a\}\rangle_G \simeq F_7$. Hence $s_a x_a \notin E(G)$ and, similarly, $s_a x_b \notin E(G)$. But then $\langle\{x_a, z_a, t_b, x_b, v_{\ell'}, t_a, s_a\}\rangle_G \simeq F_6$. This contradiction completes the proof in Subcase 2.1.

<u>Subcase 2.2:</u> R contains no P_3 both endvertices of which are leaves of R.

In this subcase R contains a P_5 (but no P_7). Using Claim 6, it is easy to show (by induction, starting with a P_5) that R is isomorphic to the subdivision of a star with center and leaves in $R \cap T$ and with vertices of degree 2 in $R \cap S$. Choose the notation such that v_{i_j} is adjacent to the center t_0 and to the leaf t_j, $j = 1, \ldots, p$. By Claim 2, there is a vertex $z_1 \in V(H_{i_1})$ such that $z_1 t_1 \in E(G)$, but $z_1 v_{i_1} \notin E(G)$. Clearly, $z_1 \notin S$; thus, by Claim 1, $z_1 v_\ell \in E(G)$ for some $v_\ell \in S$ with $i_1 < \ell \leq m$. Note that v_ℓ is not adjacent to any of v_{i_1}, t_1, but possibly $t_0 v_\ell \in E(G)$.

Suppose first that $z_1 t_j \in E(G)$ for some j, $2 \leq j \leq p$. Then clearly also $v_\ell v_{i_j}, v_\ell t_j \notin E(G)$. We further have $t_0 z_1 \notin E(G)$ (since otherwise $\langle\{z_1, t_1, v_{i_1}, t_0, t_j\}\rangle_G \simeq F_3$) and $v_{i_j} z_1 \notin E(G)$ (otherwise $\langle\{v_{i_j}, z_1, t_1, v_{i_1}, t_0, t_j\}\rangle_G \simeq F_5$). Now we have $t_0 v_\ell \in E(G)$, since otherwise $\langle\{z_1, t_1, v_{i_1}, t_0, v_{i_j}, t_j, v_\ell\}\rangle_G \simeq F_7$. This implies $v_\ell \in R \cap S$ and, by the structure of R, v_ℓ has a (unique) neighbor t_ℓ in $R \cap T$. But now $\langle\{t_\ell, v_\ell, z_1, t_1, v_{i_1}, t_0, v_{i_j}\}\rangle_G \simeq F_6$ if $t_\ell z_1 \notin E(G)$, or $\langle\{v_\ell, z_1, t_1, v_{i_1}, t_0, t_\ell\}\rangle_G \simeq F_5$, if $t_\ell z_1 \in E(G)$, respectively. This contradiction proves that z_1 is not adjacent to any of $t_2, \ldots, t_p$.

Now suppose that $z_1 v_{i_a} \notin E(G)$ for some a, $2 \leq a \leq p$. Then $t_0 z_1 \notin E(G)$ (otherwise $\langle\{t_0, v_{i_1}, t_1, z_1, v_{i_a}\}\rangle_G \simeq F_3$) and $t_0 v_\ell \in E(G)$ (otherwise $\langle\{v_\ell, z_1, t_1, v_{i_1}, t_0, v_{i_a}, t_a\}\rangle_G \simeq F_1$). This implies, as before, that v_ℓ is in $R \cap S$ and has a (unique) neighbor t_ℓ in $R \cap T$, but then $\langle\{v_\ell, t_0, v_{i_1}, t_1, z_1, v_{i_a}, t_\ell\}\rangle_G \simeq F_6$. Hence z_1 is adjacent to all vertices in $(R \cap S) \setminus \{v_{i_1}\}$. This immediately implies $p = |R \cap S| = 2$, for otherwise $\langle\{t_0, v_{i_2}, z_1, v_{i_3}, v_{i_1}\}\rangle_G \simeq F_3$.

Summarizing, it remains to consider the case when $R \cap S = \{v_{i_1}, v_{i_2}\}$, $R \cap T = \{t_0, t_1, t_2\}$ and $N_R(z_1) = \{t_1, v_{i_2}\}$. We consider the graph H_{i_2}. Since $i_1 < i_2$, $\{v_{i_1}, t_0, t_1\} \cap V(H_{i_2}) = \emptyset$, and since $z_1 v_{i_1}, t_2 v_{i_1} \notin E(G)$, we have $z_1, t_2 \in V(H_{i_2})$. By Claim 2 (for $j = i_2$), there is a vertex $z_1' \in V(H_{i_2})$ such that $z_1' z_1 \in E(G)$ but $z_1' v_{i_2} \notin E(G)$ (and, of course, $z_1' v_{i_1} \notin E(G)$). If $t_0 z_1' \notin E(G)$, then for $t_1 z_1' \in E(G)$ we have $\langle\{t_1, z_1, v_{i_1}, t_0, v_{i_2}, z_1'\}\rangle_G \simeq F_5$, and for $t_1 z_1' \notin E(G)$ we have $\langle\{z_1, t_1, v_{i_1}, t_0, v_{i_2}, z_1', t_2\}\rangle_G \simeq F_6$ if $z_1' t_2 \notin E(G)$ and $\langle\{v_{i_2}, t_2, z_1', z_1, t_0\}\rangle_G \simeq F_3$ if $z_1' t_2 \in E(G)$. Hence $t_0 z_1' \in E(G)$, but this implies $\langle\{t_0, v_{i_2}, z_1, z_1', v_{i_1}\}\rangle_G \simeq F_3$. This final contradiction completes the proof. ∎

Acknowledgement. The authors thank to an anonymous reader for pointing out a small gap in the proof of Theorem 1.

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