Closure and forbidden pairs for hamiltonicity

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Abstract

Let C be the claw $K_{1,3}$ and N the net, i.e. the only connected graph with degree sequence 333111. It is known [Bedrossian 1991; Faudree and Gould 1997] that if X, Y is a pair of connected graphs, then, for any 2-connected graph G, G being XY-free implies G is hamiltonian if and only if X is the claw C and Y belongs to a finite list of graphs, one of them being the net N.

For any such pair X, Y we show that the closures of all 2-connected XY-free graphs form a subclass of the class of CN-free graphs, and we fully describe their structure.

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1 Introduction

We consider finite simple undirected graphs $G = (V(G), E(G))$. For concepts and notation not defined here we refer the reader to [2].

We denote by $c(G)$ the *circumference* of G, i.e. the length of a longest cycle in G, by $N_G(x)$ the *neighborhood* of a vertex x in G, and we denote $N_G[x] = N_G(x) \cup \{x\}$. For a set $A \subset V(G)$, the *induced subgraph on* A is denoted by $\langle A \rangle_G$. Similarly, for a set $B \subset E(G)$, the (not necessarily induced) subgraph of G with edge set B and with the corresponding edge set is denoted by $\langle B \rangle_G$. For a set $A \subset V(G)$, the notation $G - A$ stands for $\langle V(G) \setminus A \rangle_G$ and we set $N_G(A) = \{x \in V(G) |\ N(x) \cap A \neq \emptyset\}$ and $N_G[A] = N_G(A) \cup A$. For a subgraph $X \subset G$ we denote $N_G(X) = N_G(V(X))$ and $N_G[X] = N_G[V(X)]$. If X, Y are graphs, then we say that a graph G is X*-free* (XY*-free*), if G does not contain a copy of the graph X (a copy of either of the graphs X, Y) as an induced subgraph. The graphs X, Y will be referred to in this context as *forbidden induced subgraphs*. In the special case $X = K_{1,3}$ we say that G is *claw-free*. Other graphs that will be often used as forbidden induced subgraphs are shown in Figure 1. Whenever we list the vertices of an induced subgraph X, the vertices are always ordered such that their degrees (in X) form a nonincreasing sequence.

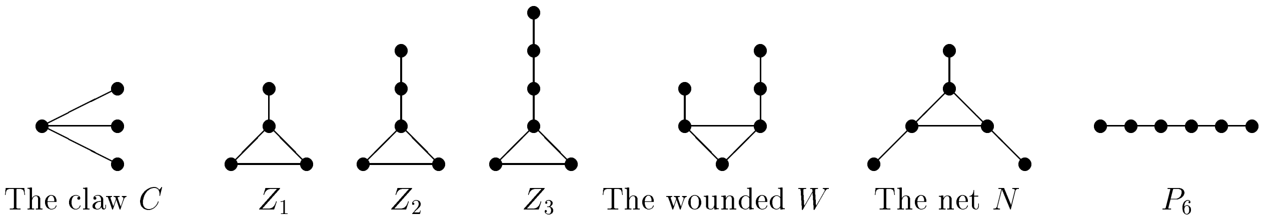


Figure 1

The following results were proved by Goodman and Hedetniemi [9], Duffus, Jacobson and Gould [6], Gould and Jacobson [10], and by Broersma and Veldman [3].

Theorem A.

(*i*) [9] *Every 2-connected CZ_1-free graph is hamiltonian.*

(*ii*) [6] *Every 2-connected CN-free graph is hamiltonian.*

(*iii*) [10] *Every 2-connected CZ_2-free graph is hamiltonian.*

(*iv*) [3] *Every 2-connected CP_6-free graph is hamiltonian.*

Bedrossian [1] characterized all pairs of forbidden subgraphs for hamiltonicity.

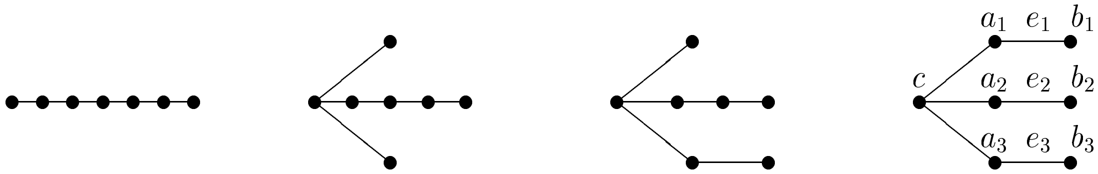
Theorem B [1]. *Let X, Y be connected graphs with $X, Y \not\simeq P_3$ and let G be a 2-connected graph that is not a cycle. Then, G being XY-free implies G is hamiltonian if and only if (up to a symmetry) $X = C$ and Y is an induced subgraph of at least one of the graphs P_6, Z_2, W or N.*

Since it was shown in [8] that the graphs in Figure 2 are the only two 2-connected nonhamiltonian CZ_3-free graphs, Theorem B was reconsidered by Faudree and Gould [7] (when the proof of the 'only if' part is now based on infinite families of graphs).

Figure 2

Theorem C [7]. *Let X, Y be connected graphs with $X, Y \not\simeq P_3$ and let G be a 2-connected graph of order $n \geq 10$ that is not a cycle. Then, G being XY-free implies G is hamiltonian if and only if (up to a symmetry) $X = C$ and Y is an induced subgraph of at least one of the graphs P_6, Z_3, W or N.*

The *line graph* of a graph H is denoted by $L(H)$. If $G = L(H)$, then we also say that H is the *line graph preimage* of G and write $H = L^{-1}(G)$. It is well-known that for any line graph $G \not\simeq K_3$ its line graph preimage is uniquely determined, and that G is k-connected ($k \geq 1$) if and only if $H = L^{-1}(G)$ is essentially k-edge-connected (i.e., every edge cut M of H such that at least two components of $H - M$ are not edgeless must contain at least k edges, or, equivalently, for any two vertex-disjoint edges $e_1 = u_1v_1$, $e_2 = u_2v_2$ of H there are k edge-disjoint paths from u_1 or v_1 to u_2 or v_2 in H). It is also easy to observe that G contains an induced subgraph isomorphic to a graph X if and only if $L^{-1}(G)$ contains a subgraph (not necessarily induced) isomorphic to $L^{-1}(X)$. The preimages of some of the graphs of Figure 1 are shown in Figure 3. When referring to the graph $L^{-1}(N)$, we will always keep the labelling of its vertices as in Figure 3.



$P_7 = L^{-1}(P_6)$ $\quad$ $L^{-1}(Z_3)$ $\quad$ $L^{-1}(W)$ $\quad$ $L^{-1}(N)$

Figure 3

For a vertex $x \in V(G)$, set $B_x = \{uv|\ u, v \in N(x), uv \notin E(G)\}$ and $G'_x = (V(G), E(G) \cup B_x)$. The graph G'_x is called the *local completion of G at x*. It was proved in [11] that if G is claw-free, then so is G'_x, and if $x \in V(G)$ is a *locally connected vertex* (i.e., $\langle N(x) \rangle_G$ is a connected graph), then $c(G) = c(G'_x)$. A vertex with connected noncomplete neighborhood is called *eligible* (in G) and the set of all eligible vertices of G is denoted by $V_{EL}(G)$.

We say that a graph F is a *closure of* G, denoted $F = \mathrm{cl}(G)$ (see [11]), if $V_{EL}(F) = \emptyset$ and there is a sequence of graphs $G_1, \ldots, G_t$ and vertices $x_1, \ldots, x_{t-1}$ such that $G_1 = G$, $G_t = F$, $x_i \in V_{EL}(G)$ and $G_{i+1} = (G_i)'_x$, $i = 1, \ldots, t-1$ (equivalently, $\mathrm{cl}(G)$ is obtained from G by a series of local completions, as long as this is possible). It was proved in [11] that

(i) the closure $\mathrm{cl}(G)$ is well-defined (i.e., uniquely determined),

(ii) there is a triangle-free graph H such that $\mathrm{cl}(G) = L(H)$,

(iii) $c(G) = c(\mathrm{cl}(G))$.

Consequently, a claw-free graph G is hamiltonian if and only if its closure $\mathrm{cl}(G)$ is too. A claw-free graph G for which $G = \mathrm{cl}(G)$ will be called *closed*. Clearly, G is closed if and only if $V_{EL}(G) = \emptyset$, i.e. every vertex $x \in V(G)$ is either *simplicial* ($\langle N(x)\rangle_G$ is a clique), or is *locally disconnected* ($\langle N(x)\rangle_G$ is disconnected, implying that, since G is claw-free, $\langle N(x)\rangle_G$ consists of two vertex disjoint cliques).

It is easy to see that if G is k-connected ($k \geq 1$) then so is $\mathrm{cl}(G)$. In Theorem 4 of [5], a characterization was given of all connected graphs X for which G being CX-free implies that $\mathrm{cl}(G)$ is also CX-free (such a CX-free class is called a *stable class*). From this characterization it follows that, among the graphs Y of Theorem C, the class of CY-free graphs is stable for $Y \in \{P_6, Z_3, N\}$, but not for $Y = W$.

In the main results of this paper, Theorems 6 and 8, we show that for any pair of graphs X, Y of Theorem C, the closure of any 2-connected X, Y-free graph is CN-free (with one simple class of exceptions) and has a very simple structure. These results are further extended in Section 4, Theorem 9, by using a recently introduced strengthening of the closure concept.

2 Closures of 2-connected CX-free graphs are CN-free

We begin with the case of the class of CP_6-free graphs.

Theorem 1. *Let G be a 2-connected graph. If G is CP_6-free, then $\mathrm{cl}(G)$ is CN-free.*

Proof. Suppose there is a 2-connected CP_6-free graph G such that $\mathrm{cl}(G)$ contains an induced N. Since G being CP_6-free implies $\mathrm{cl}(G)$ is also CP_6-free (see [5]), we can suppose that G is closed. Let $H = L^{-1}(G)$. Then H contains a (not necessarily induced) subgraph T isomorphic to $L^{-1}(N)$ (with the labelling of vertices and edges as in Fig. 3). We show that H contains a copy of $L^{-1}(P_6) = P_7$.

Since G is 2-connected, H is essentially 2-edge-connected. By symmetry, we can suppose that H contains an e_1, e_2-path P that is edge-disjoint from the path a_1ca_2 and does not contain either of the vertices a_3, b_3. Let $P = d_0d_1\ldots d_k$ with $k \geq 1$, $d_0 \in \{a_1, b_1\}$ and $d_k \in \{a_2, b_2\}$.

Suppose first that $d_0 = b_1$. If $c \notin V(P)$, then for $d_k = a_2$ the path $b_2d_kd_{k-1}\ldots d_0a_1ca_3b_3$ and for $d_k = b_2$ the path $a_1d_0d_1\ldots d_ka_2ca_3b_3$ contains a P_7. Hence $c \in V(P)$. Since H is triangle-free, $c \neq d_1$. Similarly, $c \neq d_{k-1}$ if $d_k = b_2$ and $c \notin \{d_{k-2}, d_{k-1}\}$ if $d_k = a_2$. But then $d_1d_0a_1ca_2d_kd_{k-1}$ (if $d_k = b_2$) or $d_1d_0a_1cd_kd_{k-1}d_{k-2}$ (if $d_k = a_2$) is a P_7. Hence we have $d_0 = a_1$ and, by symmetry, $d_k = a_2$. This immediately implies $k \geq 2$ since H is triangle-free. If $c \notin V(P)$, then $b_3a_3cd_0d_1\ldots d_kb_2$ contains a P_7, hence $c \in V(P)$. Then analogously $k \geq 6$ and $c \notin \{d_1, d_2, d_{k-2}, d_{k-1}\}$ since H is triangle-free, but then $d_2d_1d_0cd_kd_{k-1}d_{k-2}$ is a P_7. This contradiction completes the proof. ■

The next theorem gives an analogous result for the class of CZ_3-free graphs. The basic idea of its proof is similar to that of Theorem 1, but more complicated since in the case of CZ_3-free graphs several small exceptions are possible (these are avoided by the assumption on the order of G) and there is also one infinite class of exceptions.

Let $\mathcal{C}^{Z_3}$ be the class of graphs obtained by identifying the endvertices of $k \geq 3$ copies of a P_4 with $2k$ distinct vertices of a clique of order at least $2k$ (see Fig. 4).

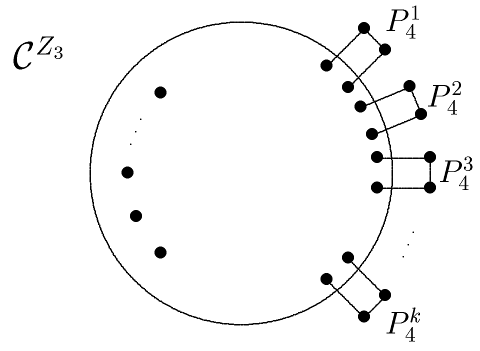


Figure 4

Theorem 2. *Let G be a 2-connected graph of order $n \geq 11$. If G is CZ_3-free, then* $\mathrm{cl}(G)$ *is CN-free or* $\mathrm{cl}(G) \in \mathcal{C}^{Z_3}$.

Proof. Proof of Theorem 2 is lengthy and is therefore postponed to Section 5.

Remark. The graph in Fig. 5a shows that the assumption $n \geq 11$ in Theorem 2 is sharp.

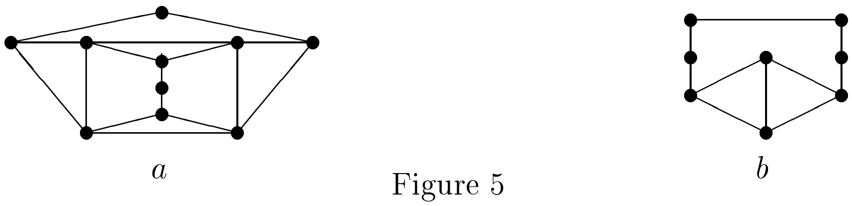


Figure 5

The situation with CW-free graphs is different due to the fact that, as already noted, G being CW-free does not imply that $\mathrm{cl}(G)$ is CW-free. An example is shown in Fig. 5b. The following two propositions help to deal with the induced W's that can possibly appear during the process of constructing $\mathrm{cl}(G)$. We denote by E, S_1 and S_2 the graphs shown in Figure 6.

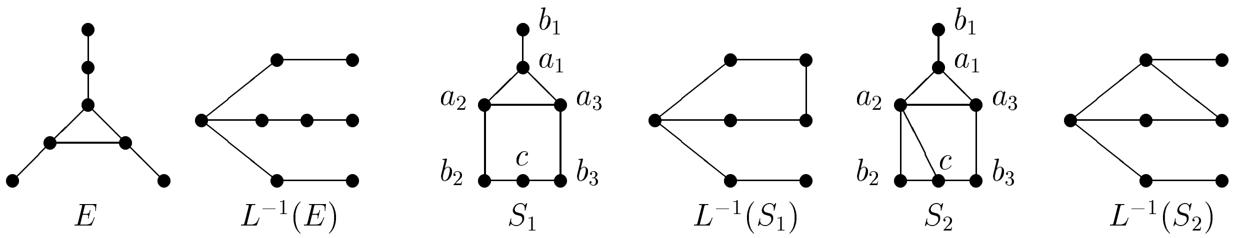


Figure 6

Proposition 3. *Let G be a CW-free graph. Then* $\mathrm{cl}(G)$ *does not contain an induced subgraph isomorphic to any of the graphs E, S_1, S_2.*

Proof. Suppose G is CW-free and let $G = G_1, G_2, \ldots, G_t = \mathrm{cl}(G)$ be the sequence of graphs that yields $\mathrm{cl}(G)$.

Claim 1. If some G_j, $1 < j \leq t$, contains an induced E, then G_{j-1} contains an induced E.

Proof follows immediately from [5], Theorem 4. □

Claim 2. If G_j contains an induced S_1 for some j, $1 < j \leq t$, then G_{j-1} contains an induced E.

Proof. Let H be an induced S_1 in G_j (with the labelling of vertices as in Fig. 6). We can suppose $|E(H) \cap B_{x_{j-1}}| \geq 1$, where the x_i's are as in the definition of closure (otherwise we are done). Suppose that $|E(H) \cap B_{x_{j-1}}| \geq 2$. Then, since $\langle N_{G_j}(x_{j-1})\rangle_{G_j}$ is a clique, $N_{G_{j-1}}(x_{j-1}) \cap V(H) = \{a_1, a_2, a_3\}$, implying $|E(H) \cap B_{x_{j-1}}| = 2$ (otherwise $\langle \{x_{j-1}, a_1, a_2, a_3\}\rangle_{G_{j-1}}$ is a claw). Thus, by symmetry, either $E(H) \cap B_{x_{j-1}} = \{a_1a_2, a_2a_3\}$, or $E(H) \cap B_{x_{j-1}} = \{a_1a_2, a_1a_3\}$. Then either $\langle \{a_3, x_{j-1}, a_1, b_3, c, a_2, b_1\}\rangle_{G_{j-1}} \simeq E$ or $\langle \{x_{j-1}, a_2, a_3, a_1, b_1, b_2, b_3\}\rangle_{G_{j-1}} \simeq E$. Hence $|E(H) \cap B_{x_{j-1}}| = 1$. Then, up to symmetry, $a_1b_1 \in B_{x_{j-1}}$, $a_2b_2 \in B_{x_{j-1}}$ or $b_2c \in B_{x_{j-1}}$, implying that $\langle \{a_1, a_2, a_3, x_{j-1}, b_1, b_2, b_3\}\rangle_{G_{j-1}} \simeq E$, $\langle \{a_3, a_1, a_2, b_3, c, b_1, x_{j-1}\}\rangle_{G_{j-1}} \simeq E$ or $\langle \{a_3, a_1, a_2, b_3, c, b_1, b_2\}\rangle_{G_{j-1}} \simeq E$. □

Claim 3. If G_j contains an induced S_2 for some j, $1 < j \leq t$, then G_{j-1} contains an induced E or S_1.

Proof. Let H be an induced S_2 in G_j. Arguing as above, $1 \leq |E(H) \cap B_{x_{j-1}}| \leq 2$. First observe that if $|E(H) \cap B_{x_{j-1}}| = 1$, then $a_1a_2 \notin B_{x_{j-1}}$ (since otherwise $\langle \{a_3, a_1, a_2, b_3\}\rangle_{G_{j-1}} \simeq C$) and, analogously, $a_1a_3, a_2a_3, a_2b_2, b_2c \notin B_{x_{j-1}}$, and that if $|E(H) \cap B_{x_{j-1}}| = 2$, then both these edges are in one of the two triangles of H. Hence it remains to consider, up to symmetry, the following possibilities.

Case $|E(H) \cap B_{x_{j-1}}| = 1$

Subcase	Induced subgraph
$a_1b_1 \in B_{x_{j-1}}$	$\langle \{a_1, a_2, a_3, x_{j-1}, b_1, b_2, b_3\}\rangle_{G_{j-1}} \simeq E$
$a_3b_3 \in B_{x_{j-1}}$	$\langle \{a_3, a_1, a_2, x_{j-1}, b_3, b_1, b_2\}\rangle_{G_{j-1}} \simeq E$
$b_3c \in B_{x_{j-1}}$	$\langle \{a_3, a_1, a_2, b_3, x_{j-1}, b_1, b_2\}\rangle_{G_{j-1}} \simeq E$
$a_2c \in B_{x_{j-1}}$	$\langle \{a_3, a_1, a_2, b_2, c, b_3, b_1\}\rangle_{G_{j-1}} \simeq S_1$

Case $|E(H) \cap B_{x_{j-1}}| = 2$

Subcase	Induced subgraph
$a_1a_2, a_1a_3 \in B_{x_{j-1}}$	$\langle \{x_{j-1}, a_2, a_3, a_1, b_1, b_2, b_3\}\rangle_{G_{j-1}} \simeq E$
$a_1a_3, a_2a_3 \in B_{x_{j-1}}$	$\langle \{x_{j-1}, a_1, a_2, a_3, b_3, b_1, b_2\}\rangle_{G_{j-1}} \simeq E$
$a_1a_2, a_2a_3 \in B_{x_{j-1}}$	$\langle \{x_{j-1}, a_1, a_3, a_2, b_2, b_1, b_3\}\rangle_{G_{j-1}} \simeq E$
$a_2b_2, b_2c \in B_{x_{j-1}}$	$\langle \{a_2, a_1, a_3, x_{j-1}, b_2, b_1, b_3\}\rangle_{G_{j-1}} \simeq E$
$a_2b_2, a_2c \in B_{x_{j-1}}$	$\langle \{a_2, a_1, a_3, x_{j-1}, b_2, b_1, b_3\}\rangle_{G_{j-1}} \simeq E$
$a_2c, b_2c \in B_{x_{j-1}}$	$\langle \{a_3, a_1, a_2, x_{j-1}, c, b_3, b_1\}\rangle_{G_{j-1}} \simeq S_1$

It is clear that, since H is induced and $\langle N_{G_j}(x_{j-1})\rangle_{G_j}$ is a clique, all these subgraphs are induced in G_{j-1}. □

Now we can complete the proof of Proposition 3. If $\mathrm{cl}(G)$ contains an induced E, S_1 or S_2, then, by Claims 1–3 and by induction, so does G. Since each of the graphs E, S_1, S_2 contains an induced W, G is not W-free, a contradiction. ■

Proposition 4. *Let G be a 2-connected closed claw-free graph. If G contains an induced N, then G contains an induced E, S_1 or S_2.*

Proof. Let $H = L^{-1}(G)$ and let $T = L^{-1}(N)$ and $P = d_0 d_1 \dots d_k$ be the same as in the proof of Theorem 1. We show that in each of the possible cases we find in H a (not necessarily induced) subgraph T' isomorphic to $L^{-1}(E)$, $L^{-1}(S_1)$ or $L^{-1}(S_2)$.

If $c = d_i$ for some i, then, since H is triangle-free, we have $i \geq 3$ for $d_0 = a_1$ and $i \geq 2$ for $d_0 = b_1$, respectively.

Case	Subgraph T'
$d_0 = a_1$	$\langle \{cd_0, d_0d_1, d_1d_2, ca_2, a_2b_2, ca_3, a_3b_3\} \rangle_H \simeq L^{-1}(E)$
$d_0 = b_1$	$\langle \{ca_1, a_1d_0, d_0d_1, ca_2, a_2b_2, ca_3, a_3b_3\} \rangle_H \simeq L^{-1}(E)$

Hence $c \notin V(P)$. Then we have, up to symmetry and since H is triangle-free, the following possibilities.

Case	Subgraph T'
$d_0 = a_1, d_k = a_2, k = 2$	$\langle \{ca_1, a_1d_1, d_1a_2, a_2c, ca_3, a_3b_3, a_1b_1\} \rangle_H \simeq L^{-1}(S_2)$
$d_0 = a_1, d_k = a_2, k > 2$	$\langle \{ca_1, a_1d_1, d_1d_2, ca_2, a_2b_2, ca_3, a_3b_3\} \rangle_H \simeq L^{-1}(E)$
$d_0 = a_1, d_k = b_2, k = 1$	$\langle \{ca_1, a_1b_2, b_2a_2, a_2c, ca_3, a_3b_3, a_1b_1\} \rangle_H \simeq L^{-1}(S_2)$
$d_0 = a_1, d_k = b_2, k > 1$	$\langle \{ca_2, a_2b_2, b_2d_{k-1}, ca_1, a_1b_1, ca_3, a_3b_3\} \rangle_H \simeq L^{-1}(E)$
$d_0 = b_1, d_k = b_2, k = 1$	$\langle \{ca_1, a_1b_1, b_1b_2, b_2a_2, a_2c, ca_3, a_3b_3\} \rangle_H \simeq L^{-1}(S_1)$
$d_0 = b_1, d_k = b_2, k > 1$	$\langle \{ca_1, a_1b_1, b_1d_1, ca_2, a_2b_2, ca_3, a_3b_3\} \rangle_H \simeq L^{-1}(E)$

This completes the proof of Proposition 4. ■

Now we can prove the following result which gives a CW-free analogue of Theorems 1 and 2.

Theorem 5. *Let G be a 2-connected graph. If G is CW-free, then* $\mathrm{cl}(G)$ *is CN-free.*

Proof. If $\mathrm{cl}(G)$ contains an induced N, then, by Proposition 4, $\mathrm{cl}(G)$ contains an induced E, S_1 or S_2, contradicting Proposition 3. ■

The results of Section 2 can now be summarized as follows.

Theorem 6. *Let G be a 2-connected graph of order $n \geq 11$. If G is CX-free for $X \in \{P_6, Z_3, W, N\}$, then either* $\mathrm{cl}(G)$ *is CN-free or* $\mathrm{cl}(G) \in \mathcal{C}^{Z_3}$. ■

3 Structure of closed CN-free graphs

In this section we describe the structure of all 2-connected closed CN-free graphs. Denote by $\mathcal{C}_1^N$ the class of graphs obtained by the following construction (see Fig. 7a).

(i) Take $k \geq 1$ complete graphs $K_1, \dots, K_k$ with $|V(K_i)| \geq 4$ for $2 \leq i \leq k-1$ (if $k \geq 3$) and $|V(K_i)| \geq 2$ for $i = 1, k$.

(ii) Choose subsets $K_1^2 \subset V(K_1)$ and $K_k^1 \subset V(K_k)$ such that $|K_1^2| \geq 2$ and $|K_k^1| \geq 2$.

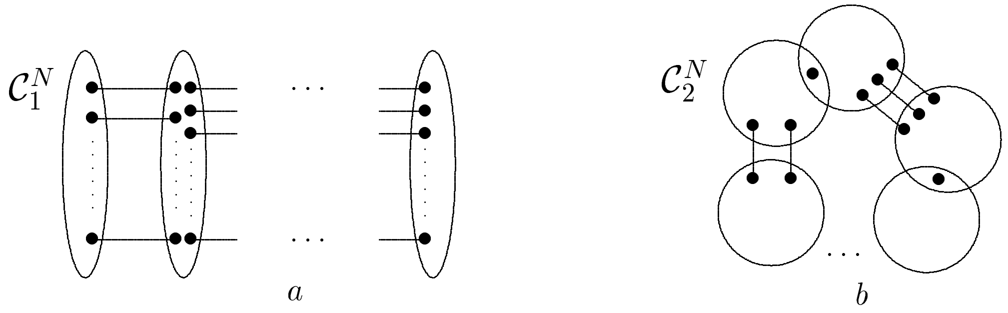


Figure 7

(iii) In each of the K_i's, $2 \leq i \leq k-1$, choose disjoint subsets $K_i^1, K_i^2 \subset V(K_i)$ such that $|K_i^1| \geq 2$, $|K_i^2| \geq 2$ and $|K_i^2| = |K_{i+1}^1|$ for every $i = 1, \dots, k-1$.

(iv) For every $i = 1, \dots, k-1$ join the vertices of K_i^2 and K_{i+1}^1 with a matching.

Further denote by $\mathcal{C}_2^N$ the class of graphs obtained by the following construction (see Fig. 7b).

(i) Take $k \geq 3$ complete graphs $K_1, \dots, K_k$ with $|V(K_i)| \geq 2$ for $i = 1, \dots, k$.

(ii) In each of the K_i's choose nonempty disjoint subsets $K_i^1, K_i^2 \subset V(K_i)$ such that $|K_i^2| = |K_{i+1}^1|$, $i = 1, \dots, k-1$, and $|K_k^2| = |K_1^1|$ (for $k = 3$ furthermore $|K_i^1| \geq 2$ for at least one i, $1 \leq i \leq 3$).

(iii) For every $i = 1, \dots, k$ identify K_i^2 with K_{i+1}^1 if $|K_i^2| = |K_{i+1}^1| = 1$ and join the vertices of K_i^2 and K_{i+1}^1 with a matching if $|K_i^2| = |K_{i+1}^1| \geq 2$ (indices modulo k), respectively.

Theorem 7. *Let G be a graph of order $n \geq 10$. Then G is a 2-connected closed CN-free graph if and only if $G \in \mathcal{C}_1^N \cup \mathcal{C}_2^N$.*

Proof. It is straightforward to check that every graph in $\mathcal{C}_1^N \cup \mathcal{C}_2^N$ is 2-connected, closed and CN-free. Let, conversely, G be a 2-connected closed CN-free graph and let $H = L^{-1}(G)$. We distinguish two cases.

Case 1: $c(H) \geq 5$.

Let C be a longest cycle in H. Suppose first that C has a chord xy. Since H is triangle-free, $|V(C)| \geq 6$. If $|V(C)| \geq 7$, we can choose an orientation of C such that the segment xCy of C has least three interior vertices, but then $\langle\{xy, yy^-, xx^+, x^+x^{++}, xx^-, x^-x^{--}\}\rangle_H \simeq L^{-1}(N)$, a contradiction. If $|V(C)| = 6$, then, since H is triangle-free and $|E(H)| = |V(G)| \geq 10$, there is an edge uv with $u \in V(C)$ and $v \notin V(C)$. Up to symmetry, we can suppose that $u = x$ or $u = x^+$, but then again either $\langle\{yx, xv, yy^-, y^-y^{--}, yy^+, y^+y^{++}\}\rangle_H \simeq L^{-1}(N)$ or $\langle\{xx^+, x^+v, xy, yy^-, xx^-, x^-x^{--}\}\rangle_H \simeq L^{-1}(N)$, respectively. Hence C is chordless.

If all neighbors outside C of the vertices on C are of degree 1, then clearly $L(H) \in \mathcal{C}_2^N$. Hence we can suppose that there are vertices x, y, z such that $xy, yz \in E(H)$, $x \in V(C)$ and $y \notin V(C)$. Clearly $z \notin \{x^-, x^+\}$ (since H is triangle-free). If $z \notin V(C)$ or $z \in V(C) \setminus \{x^{--}, x^{++}\}$, then $\langle\{xy, yz, xx^-, x^-x^{--}, xx^+, x^+x^{++}\}\rangle_H \simeq L^{-1}(N)$; hence $z \in \{x^{--}, x^{++}\}$. By symmetry, let $z = x^{++}$.

Suppose y has another neighbor u. Then $u \notin V(C)$ (otherwise we have a longest cycle with a chord) and $\langle\{xy, yu, xx^+, x^+x^{++}, xx^-, x^-x^{--}\}\rangle_H \simeq L^{-1}(N)$. Hence the vertex y, and indeed all common neighbors of x and x^{++}, are of degree 2. This fact together with a straightforward inductive argument shows that $L(H) = G \in \mathcal{C}_2^N$.

<u>Case 2:</u> $c(H) = 4$.

Let $P = d_0d_1 \dots d_\ell$ be a diameter path in H (i.e. a shortest path joining two vertices at maximum distance in H). Suppose first that $\ell \geq 5$. Since H is essentially 2-edge-connected, d_1d_2 cannot be a cut-edge. If d_1 and d_2 have adjacent neighbors u_1 and u_2, respectively, then $\langle\{d_2u_2, u_2u_1, d_2d_1, d_1d_0, d_2d_3, d_3d_4\}\rangle_H \simeq L^{-1}(N)$. Hence $|N(d_0) \cap N(d_2)| \geq 2$ or $|N(d_1) \cap N(d_3)| \geq 2$ (recall that $c(H) = 4$ and H is triangle-free). Now, if some $x \in N(d_0) \cap N(d_2)$ has another neighbor z, then $z \notin V(P)$ (since P is a diameter path) and $\langle\{d_2x, xz, d_2d_1, d_1d_0, d_2d_3, d_3d_4\}\rangle_H \simeq L^{-1}(N)$. Hence in the first case all common neighbors of d_0 and d_2 (or, analogously, in the second case of d_1 and d_3) are of degree 2. A straightforward inductive argument then gives $G = L(H) \in \mathcal{C}_1^N$.

Let next $\ell = 4$. Arguing as above, for $|N(d_0) \cap N(d_2)| \geq 2$ we get $G = L(H) \in \mathcal{C}_1^N$. Let thus $x \in N(d_1) \cap N(d_3)$, $x \neq d_2$. If $|N(d_1) \cap N(d_3)| = 2$, then all neighbors of x and d_2 outside P are of degree 1 (otherwise we have an $L^{-1}(N)$), implying $G = L(H) \in \mathcal{C}_2^N$ (with $k = 4$); if $|N(d_1) \cap N(d_3)| \geq 3$, then moreover at most one common neighbor of d_1, d_3 can have some further neighbors of degree 1 (otherwise we have an $L^{-1}(N)$), implying again $G = L(H) \in \mathcal{C}_2^N$ (with $k = 3$).

The cases $\ell = 2, 3$ are trivial. ■

Combining Theorems 6 and 7, we now have the following result.

Theorem 8. *Let G be a 2-connected graph of order $n \geq 11$. If G is CX-free for $X \in \{P_6, Z_3, W, N\}$, then $\mathrm{cl}(G) \in \mathcal{C}^{Z_3} \cup \mathcal{C}_1^N \cup \mathcal{C}_2^N$.* ■

4 Strong closure

In [4], the closure concept was strengthened in the following way.

Let G be a closed claw-free graph and let $H = L^{-1}(G)$. A k-cycle C in G is said to be *eligible* if $4 \leq k \leq 6$ and at least $k-3$ nonconsecutive edges of C are contained in no clique of order at least 3 (or, equivalently, if the k-cycle $L^{-1}(C)$ in H contains at least $k-3$ nonconsecutive vertices of degree 2).

For an eligible cycle C in G set $B'_C = \{uv|\ u, v \in N_G[C], u, v \notin E(G)\}$. The graph $G'_C = (V(G), E(G) \cup B'_C)$ is called the *cycle-completion of G at C*.

Let now G be a claw-free graph. A graph $\mathrm{cl}_C(G)$ is said to be a *cycle closure* of G, if there is a sequence of graphs $G_1, \dots, G_t$ such that

(*i*) $G_1 = \mathrm{cl}(G)$,

(*ii*) $G_{i+1} = \mathrm{cl}((G_i)'_C)$ for some eligible cycle C in G_i, $i = 1, \dots, t-1$,

(*iii*) $G_t = \mathrm{cl}_C(G)$ contains no eligible cycle.

Thus, $\mathrm{cl}_C(G)$ is obtained from $\mathrm{cl}(G)$ by recursively performing the cycle-completion operation at eligible cycles and then closing the resulting graph with the (obvious) closure, as long as this is possible. The following result was proved in [4].

Theorem D [4]. *Let G be a claw-free graph. Then*
 (i) $\mathrm{cl}_C(G)$ *is uniquely determined,*
 (ii) $c(G) = c(\mathrm{cl}_C(G))$.

Let now $\mathcal{C}^N \subset \mathcal{C}_2^N$ be the subclass of all graphs from $\mathcal{C}_2^N$ for which $|K_i^1| = |K_i^2| = 1$ for all i, $1 \le i \le k$ (see Figure 8). Then it is straightforward to check that
 (i) if $G \in \mathcal{C}^{Z_3}$, then $\mathrm{cl}_C(G)$ is complete (and hence N-free),
 (ii) if $G \in \mathcal{C}_1^N$, then $\mathrm{cl}_C(G)$ is complete,
 (iii) if $G \in \mathcal{C}_2^N$, then either $\mathrm{cl}_C(G)$ is complete, or $\mathrm{cl}_C(G) \in \mathcal{C}^N$.

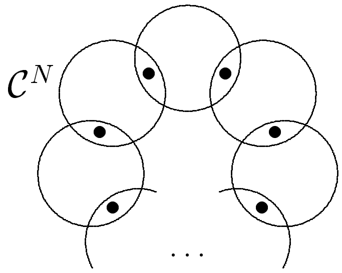


Figure 8

From Theorems 6 and 8 we then immediately have the following result.

Theorem 9. *Let G be a 2-connected graph of order $n \ge 11$. If G is CX-free for $X \in \{P_6, Z_3, W, N\}$, then its cycle closure $\mathrm{cl}_C(G)$ is either complete or belongs to $\mathcal{C}^N$.* ■

5 Proof of Theorem 2

Let, to the contrary, G be a 2-connected CZ_3-free graph such that $\mathrm{cl}(G)$ contains an induced N. Since G being CN-free implies that $\mathrm{cl}(G)$ is also CN-free (see [5]), we can suppose that G is closed. Let H, T and $P = d_0, d_1, \ldots, d_k$ be the same as in the proof of Theorem 1.

<u>Case 1:</u> $c \in V(P)$.
Then $c = d_\ell$ for some ℓ, $l \le \ell \le k-1$. If $d_0 = a_1$, then, since H is triangle-free, $\ell \ge 3$, but then $\langle\{ca_3, ca_2, cd_{\ell-1}, d_{\ell-1}d_{\ell-2}, \ldots, d_1a_1, a_1b_1\}\rangle_H$ contains an $L^{-1}(Z_3)$. Hence $d_0 \ne a_1$ and, by symmetry, $d_k \ne a_2$, implying that $d_0 = b_1$ and $d_k = b_2$.

Since H is triangle-free, $2 \le \ell \le k-2$. If $\ell \ge 3$, then $\langle\{ca_3, ca_2, ca_1, a_1b_1, b_1d_1, d_1d_2\}\rangle_H \simeq L^{-1}(Z_3)$; hence $\ell = 2$. By symmetry, $k = 4$. Relabel the vertices $d_1 := z_1$ and $d_3 := z_2$ and set $T_1 = \langle E(T) \cup \{b_1z_1, z_1c, b_2z_2, z_2c\}\rangle_H$. Since G is 2-connected, there is a path $P' = d'_0d'_1 \ldots d'_{k'}$ ($k' \ge 1$) in H such that $d'_0 \in \{a_3, b_3\}$, $d'_{k'} \in V(T_1) \setminus \{a_3, b_3\}$ and P' does not contain the edge

ca_3. Immediately $k' \geq 2$, for otherwise P' is an edge, but every such additional edge in T_1 that does not create a triangle yields an $L^{-1}(Z_3)$:

Case	Contradiction
$d'_0 = a_3, d'_1 = b_1$	$\langle\{ca_2, cz_1, ca_1, a_1b_1, b_1a_3, a_3b_3\}\rangle_H \simeq L^{-1}(Z_3)$
$d'_0 = b_3, d'_1 = b_1$	$\langle\{ca_1, ca_2, cz_1, z_1b_1, b_1b_3, b_3a_3\}\rangle_H \simeq L^{-1}(Z_3)$
$d'_0 = b_3, d'_1 = a_1$	$\langle\{ca_2, ca_3, cz_1, z_1b_1, b_1a_1, a_1b_3\}\rangle_H \simeq L^{-1}(Z_3)$

Hence $k' \geq 2$, implying that some interior vertex of P' is in $V(H) \setminus V(T_1)$.

We show that if $xy \in E(H)$ for some $x \in V(H) \setminus V(T_1)$ and $y \in V(T_1)$, then $y = b_3$ or $y = c$. Indeed, there are, up to symmetry, the following remaining subcases.

Case	Contradiction
$y = b_1$	$\langle\{b_1x, b_1z_1, b_1a_1, a_1c, ca_3, a_3b_3\}\rangle_H \simeq L^{-1}(Z_3)$
$y = a_1$	$\langle\{ca_3, ca_2, cz_1, z_1b_1, b_1a_1, a_1x\}\rangle_H \simeq L^{-1}(Z_3)$
$y = a_3$	$\langle\{a_3b_3, a_3x, a_3c, ca_1, a_1b_1, b_1z_1\}\rangle_H \simeq L^{-1}(Z_3)$

This implies that $d'_0 = b_3, d'_{k'} = c$ and, as with P, $k' = 2$. Relabel the vertex $d'_1 := z_3$ and set $T_2 = \langle E(T_1) \cup \{cz_3, z_3b_3\}\rangle_H$. By the above considerations and by symmetry, there are no more edges between the vertices of T_2 (i.e., T_2 is induced in H), and $xy \in E(H)$ for $x \in V(H) \setminus V(T_2)$ and $y \in V(T_2)$ implies $y = c$. Let thus $xc \in E(H)$. If $d_H(x) \geq 2$ and $ux \in E(H)$, $u \neq c$, then clearly $u \notin V(T_2)$ (since e.g. $u = b_1$ implies $\langle\{b_1a_1, b_1z_1, b_1x, xc, ca_2, a_2b_2\}\rangle_H \simeq L^{-1}(Z_3)$; the other cases are symmetric or yield a triangle), and then the 2-connectedness of H implies, as above, the existence of a path $P'' = d''_0d''_1 \dots d''_{k''}$ with $d''_0 = u$, $d''_{k''} = c$ and $k'' = 2$. Relabelling $x := a_4, u := b_4, d''_1 := z_4$ and setting $T_3 = \langle E(T_2) \cup \{ca_4, a_4b_4, b_4z_4, z_4c\}\rangle_H$, by a straightforward inductive argument we get that H consists of 4-cycles $\langle\{ca_i, a_ib_i, b_iz_i, z_ic\}\rangle_H$, $i = 1, \dots, s$, and of edges cx_j, $j = 1, \dots, t$, for some integers $s \geq 3$, $t \geq 0$. This implies that $G = L(H) \in \mathcal{C}^{Z_3}$.

<u>Case 2:</u> $c \notin V(P)$.

We distinguish (up to symmetry) three possible subcases.

<u>Subcase 2a:</u> $d_0 = a_1$, $d_k = a_2$.

Since H is triangle-free, $k \geq 2$. If $k \geq 3$, then $\langle\{a_1b_1, a_1c, a_1d_1, d_1d_2, \dots, d_kb_2\}\rangle_H$ contains an $L^{-1}(Z_3)$, hence $k = 2$. Set $T_1 = \langle E(T) \cup \{a_1d_1, d_1a_2\}\rangle_H$. We check that there is no edge xy with $x \in V(H) \setminus V(T_1)$ and $y \in V(T_1)$. Up to symmetry, there are the following possibilities.

Case	Contradiction
$y = b_1$	$\langle\{a_2b_2, a_2c, a_2d_1, d_1a_1, a_1b_1, b_1x\}\rangle_H \simeq L^{-1}(Z_3)$
$y = a_1$	$\langle\{a_1x, a_1b_1, a_1d_1, d_1a_2, a_2c, ca_3\}\rangle_H \simeq L^{-1}(Z_3)$
$y = c$	$\langle\{cx, ca_3, ca_1, a_1d_1, d_1a_2, a_2b_2\}\rangle_H \simeq L^{-1}(Z_3)$
$y = d_1$	$\langle\{d_1x, d_1a_2, d_1a_1, a_1c, ca_3, a_3b_3\}\rangle_H \simeq L^{-1}(Z_3)$
$y = a_3$	$\langle\{a_3x, a_3b_3, a_3c, ca_1, a_1d_1, d_1a_2\}\rangle_H \simeq L^{-1}(Z_3)$
$y = b_3$	$\langle\{a_1b_1, a_1d_1, a_1c, ca_3, a_3b_3, b_3x\}\rangle_H \simeq L^{-1}(Z_3)$

Since $n \geq 11$ and $|E(T_1)| = 8$, there are at least three further edges joining the vertices of T_1. The following edges are impossible:

Edge	Contradiction
b_1a_2	$\langle\{a_2b_2, a_2b_1, a_2d_1, d_1a_1, a_1c, ca_3\}\rangle_H \simeq L^{-1}(Z_3)$
b_1a_3	$\langle\{a_3b_3, a_3c, a_3b_1, b_1a_1, a_1d_1, d_1a_2\}\rangle_H \simeq L^{-1}(Z_3)$
b_1b_3	$\langle\{a_2d_1, a_2b_2, a_2c, ca_3, a_3b_3, b_3b_1\}\rangle_H \simeq L^{-1}(Z_3)$
a_1b_2	$\langle\{a_1b_1, a_1b_2, a_1d_1, d_1a_2, a_2c, ca_3\}\rangle_H \simeq L^{-1}(Z_3)$
a_1b_3	$\langle\{a_1b_1, a_1d_1, a_1b_3, b_3a_3, a_3c, ca_2\}\rangle_H \simeq L^{-1}(Z_3)$

Thus, it is straightforward to check that among the edges that do not create a triangle the only remaining possibilities are the edges b_1b_2, d_1a_3 and d_1b_3. Since H is triangle-free, only one of the edges d_1a_3, d_1b_3 can be present. Hence $n = |E(H)| \leq 10$, a contradiction.

<u>Subcase 2b:</u> $d_0 = b_1$, $d_k = b_2$.

For $k \geq 2$ we have $\langle\{ca_2, ca_3, ca_1, a_1b_1, b_1d_1, d_1d_2\}\rangle_H \simeq L^{-1}(Z_3)$, hence $k = 1$ (i.e. $b_1b_2 \in E(H)$). Set $T_1 = \langle E(T) \cup \{b_1b_2\}\rangle_H$. Up to symmetry, the only possible further edges that join two vertices of T_1 and do not create a triangle are the edges b_1a_3, b_1b_3 and a_1b_3. Since H is triangle-free, at most one of the edges b_1a_3, b_1b_3 can be present. Since $n \geq 11$, H contains at least two edges having at least one vertex in $R = V(H) \setminus V(T_1)$. We consider the possible edges xy with $x \in R$ and $y \in V(T_1)$:

Case	Contradiction
$y = b_1$	$\langle\{b_1x, b_1a_1, b_1b_2, b_2a_2, a_2c, ca_3\}\rangle_H \simeq L^{-1}(Z_3)$
$y = c$	$\langle\{cx, ca_3, ca_1, a_1b_1, b_1b_2, b_2a_2\}\rangle_H \simeq L^{-1}(Z_3)$
$y = a_3$	$\langle\{a_3x, a_3b_3, a_3c, ca_1, a_1b_1, b_1b_2\}\rangle_H \simeq L^{-1}(Z_3)$

Thus, up to symmetry, the only remaining possibilities are either $y = a_1$ or $y = b_3$.

If a_1 has two distinct neighbors $x_1, x_2 \in R$, then $\langle\{a_1x_1, a_1x_2, a_1b_1, b_1b_2, b_2a_2, a_2c\}\rangle_H \simeq L^{-1}(Z_3)$. Similarly, $x_1, x_2 \in N_H(b_3) \cap R$ implies $\langle\{b_3x_1, b_3x_2, b_3a_3, a_3c, ca_1, a_1b_1\}\rangle_H \simeq L^{-1}(Z_3)$. Hence a_1 and b_3 (and, by symmetry also a_2) can have at most one neighbor in R.

Next we show that at most one of a_1, a_2 can have a neighbor in R. Let thus $x_i \in N(a_i) \cap R$, $i = 1, 2$. If $x_1 = x_2$, we are in Subcase 2a; hence $x_1 \neq x_2$, but then $\langle\{a_1x_1, a_1c, a_1b_1, b_1b_2, b_2a_2, a_2x_2\}\rangle_H \simeq L^{-1}(Z_3)$, a contradiction. Hence we can suppose that $N(a_2) \cap R = \emptyset$.

If a_1 has no neighbor in R, then (since $n \geq 11$ and G is connected) there are $x_1, x_2 \in R$ such that $b_3x_1, x_1x_2 \in E(H)$, implying $\langle\{ca_1, ca_2, ca_3, a_3b_3, b_3x_1, x_1x_2\}\rangle_H \simeq L^{-1}(Z_3)$. Thus, $|N(a_1) \cap R| = 1$. Denote the (only) neighbor of a_1 in R by u.

Suppose that $ub_3 \in E(H)$. Then, since H is triangle-free, $a_1b_3 \notin E(H)$. Since $n \geq 11$ and at most one of the two remaining possible edges inside T_1, namely b_1a_3 and b_1b_3, can occur, necessarily $ux \in E(H)$ for some further $x \in R$. But then $\langle\{ux, ub_3, ua_1, a_1b_1, b_1b_2, b_2a_2\}\rangle_H \simeq L^{-1}(Z_3)$. Hence $ub_3 \notin E(H)$.

Now, if $x \in R$ is adjacent to b_3, then $x \neq u$, implying $\langle\{a_1u, a_1b_1, a_1c, ca_3, a_3b_3, b_3x\}\rangle_H \simeq L^{-1}(Z_3)$. Hence b_3 has no neighbor in R.

Since $n \geq 11$, $|N(a_1) \cap R| = 1$ and $|N(y) \cap R| = 0$ for all other $y \in V(T_1)$, there is a vertex $v \in R$ with $uv \in E(H)$. Since $u, v \in R$ and the vertices of T_1 can have no other adjacencies in R, ua_1 is a cut-edge separating the edge uv from T_1, a contradiction.

<u>Subcase 2c:</u> $d_0 = a_1$, $d_k = b_2$.

For $k \geq 3$ the subgraph $\langle\{a_1b_1, a_1c, a_1d_1, d_1d_2, \ldots, b_2a_2\}\rangle_H$ contains an $L^{-1}(Z_3)$, hence $k = 1$ or $k = 2$.

First suppose that $k = 1$, i.e. $a_1b_2 \in E(H)$. Let $T_1 = \langle E(T) \cup \{a_1b_2\}\rangle_H$ and denote $R = V(H) \setminus V(T_1)$. We again consider further possible edges joining vertices of T_1.

Edge	Contradiction
b_1a_3	$\langle\{a_3b_3, a_3c, a_3b_1, b_1a_1, a_1b_2, b_2a_2\}\rangle_H \simeq L^{-1}(Z_3)$
a_1b_3	$\langle\{a_1b_1, a_1b_2, a_1b_3, b_3a_3, a_3c, ca_2\}\rangle_H \simeq L^{-1}(Z_3)$
a_2b_3	$\langle\{a_1b_1, a_1c, a_1b_2, b_2a_2, a_2b_3, b_3a_3\}\rangle_H \simeq L^{-1}(Z_3)$

Since H is triangle-free and since the edges b_1b_3 and b_2b_3 reduce the situation to Subcase 2b, the only possible edges inside T_1 are b_1a_2 and b_2a_3. If both are present, then $\langle\{a_3b_3, a_3c, a_3b_2, b_2a_2, a_2b_1, b_1a_1\}\rangle_H \simeq L^{-1}(Z_3)$. Hence at most one of the edges b_1a_2, b_2a_3 is in $E(H)$. Since $n \geq 11$, there are at least three edges having a vertex in R.

Next we consider the edges xy with $x \in R$ and $y \in V(T_1)$.

Case	Contradiction
$y = a_1$	$\langle\{a_1x, a_1b_1, a_1b_2, b_2a_2, a_2c, ca_3\}\rangle_H \simeq L^{-1}(Z_3)$
$y = c$	$\langle\{cx, ca_3, ca_2, a_2b_2, b_2a_1, a_1b_1\}\rangle_H \simeq L^{-1}(Z_3)$
$y = b_2$	$\langle\{b_2x, b_2a_1, b_2a_2, a_2c, ca_3, a_3b_3\}\rangle_H \simeq L^{-1}(Z_3)$
$y = a_3$	$\langle\{a_3x, a_3b_3, a_3c, ca_2, a_2b_2, b_2a_1\}\rangle_H \simeq L^{-1}(Z_3)$
$y = b_3$	$\langle\{a_1b_1, a_1b_2, a_1c, ca_3, a_3b_3, b_3x\}\rangle_H \simeq L^{-1}(Z_3)$

Thus, the only possible cases are $y = b_1$ and $y = a_2$. If b_1 has two neighbors x_1, x_2 in R, then $\langle\{b_1x_1, b_1x_2, b_1a_1, a_1c, ca_3, a_3b_3\}\rangle_H \simeq L^{-1}(Z_3)$. Hence $|N(b_1) \cap R| \leq 1$ and, similarly, $|N(a_2) \cap R| \leq 1$. This implies that there are vertices $x_1, x_2 \in R$ such that either $b_1x_1, x_1x_2 \in E(H)$, or $a_2x_1, x_1x_2 \in E(H)$. But then either $\langle\{ca_3, ca_2, ca_1, a_1b_1, b_1x_1, x_1x_2\}\rangle_H \simeq L^{-1}(Z_3)$ or $\langle\{a_1c, a_1b_1, a_1b_2, b_2a_2, a_2x_1, x_1x_2\}\rangle_H \simeq L^{-1}(Z_3)$, respectively.

It remains to consider the case $k = 2$. Set $T_1 = \langle E(T) \cup \{ad_1, d_1b_2\}\rangle_H$ and $R = V(H) \setminus V(T_1)$. Now it is straightforward to check that $d_1b_3 \in E(H)$ implies $\langle\{d_1a_1, d_1b_2, d_1b_3, b_3a_3, a_3c, ca_2\}\rangle_H \simeq L^{-1}(Z_3)$, and that each of the further edges with both vertices in T_1 either creates a triangle or reduces the situation to one (or more) of the previous subcases. Considering the edges xy with $x \in R$ and $y \in V(T_1)$, we have the following.

Case	Contradiction
$y = a_1$	$\langle\{a_1x, a_1b_1, a_1d_1, d_1b_2, b_2a_2, a_2c\}\rangle_G \simeq L^{-1}(Z_3)$
$y = c$	$\langle\{cx, ca_3, ca_2, a_2b_2, b_2d_1, d_1a_1\}\rangle_G \simeq L^{-1}(Z_3)$
$y = d_1$	$\langle\{d_1x, d_1a_1, d_1b_2, b_2a_2, a_2c, ca_3\}\rangle_G \simeq L^{-1}(Z_3)$
$y = b_2$	$\langle\{b_2x, b_2d_1, b_2a_2, a_2c, ca_3, a_3b_3\}\rangle_G \simeq L^{-1}(Z_3)$
$y = a_2$	$\langle\{a_2x, a_2c, a_2b_2, b_2d_1, d_1a_1, a_1b_1\}\rangle_G \simeq L^{-1}(Z_3)$
$y = a_3$	$\langle\{a_3x, a_3b_3, a_3c, ca_2, a_2b_2, b_2d_1\}\rangle_G \simeq L^{-1}(Z_3)$
$y = b_3$	$\langle\{a_1b_1, a_1d_1, a_1c, ca_3, a_3b_3, b_3x\}\rangle_G \simeq L^{-1}(Z_3)$

Thus, the only possible case is $y = b_1$. Since $n \geq 11$, there is a vertex $x \in R$ with $xb_1 \in E(H)$. But then, since a_1 is the only neighbor of b_1 in $V(T_1)$ and since the other vertices of T_1 have no adjacencies in R, a_1b_1 is a cut-edge separating the edge b_1x from the rest of T_1. This contradiction completes the proof of Theorem 2. ■

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