Contractibility techniques as a closure concept

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Abstract

We introduce a closure concept in the class of line graphs and claw-free graphs based on contractibility of certain subgraphs in the line graph preimage. The closure can be considered as a common generalization and strengthening of the reduction techniques of Catlin and Veldman and of the closure concept introduced by the first author. We show that the closure is uniquely determined and the closure operation preserves the circumference of the graph.

Keywords: closure, contractible graph, collapsible graph, line graph, claw-free graph, hamiltonian graph, circumference

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1 Introduction

All graphs considered here are finite undirected graphs without loops. However, in some situations we allow the graphs to have multiple edges. We follow the most common graph-theoretic terminology and notation. For concepts and notation not defined here we refer the reader to [2]. Specifically, $N(x)$ and $N(M)$ denote the neighborhood of a vertex $x \in V(G)$ or a set $M \subset V(G)$, i.e. the set of all vertices that are adjacent to x (or adjacent to a vertex in M), and $d_H(x)$ denotes the degree of a vertex x in H. If F is a subgraph of a graph H, then a vertex x is said to be a *vertex of attachment of F in H* if $x \in V(F)$ and x has a neighbor in $V(H) \setminus V(F)$. The set of all vertices of attachment of a subgraph F in H is denoted by $A_H(F)$. For a set $M \subset H$, $\langle M \rangle_H$ denotes the subgraph induced by M in H.

The *line graph* of a graph H is denoted by $L(H)$. It is well-known that for every line graph G which is not a triangle there is a unique graph H such that $L(H) = G$. This graph H is called the *line graph preimage* of G and denoted by $H = L^{-1}(G)$. A *pendant edge* is an edge having a vertex of degree 1; a pendant edge corresponds to a *simplicial vertex* in $L(H)$, a vertex the neighborhood of which induces a complete graph.

A *dominating closed trail* (abbreviated DCT) in a graph H is a closed trail T such that every edge of H has at least one vertex in T. (A closed trail is defined as usual, except that we allow a single vertex to be such a trail.) Harary and Nash-Williams [5] proved the following result, relating the existence of a DCT in H to the hamiltonicity of $L(H)$.

Theorem A [5]. *Let H be a graph with at least three edges. Then $L(H)$ is hamiltonian if and only if H contains a DCT.*

In particular, if H has a spanning eulerian subgraph, then $L(H)$ is hamiltonian. Let $d_T(H)$ denote the maximum number of edges of a graph H that are dominated by a closed trail T (a *maximum closed trail*) and let $c(G)$ denote the *circumference* (the length of a longest cycle) of G. Clearly, if $G = L(H)$ is hamiltonian, then $d_T(H) = |E(H)| = |V(G)| = c(G)$. It is easy to see that in fact $d_T(H) = c(L(H))$ for any graph H with at least three edges.

A graph G is *claw-free* if G does not contain a copy of the *claw* $K_{1,3}$ as an induced subgraph. Beineke [1] characterized line graphs by showing that a graph G is a line graph (of some graph) if and only if G does not contain an induced subgraph which is isomorphic to some of nine given graphs, one of them being the claw. Thus, the class of claw-free graphs can be considered as a natural extension of the class of line graphs.

Let G be a claw-free graph. A vertex $x \in V(G)$ is *locally connected* if $\langle N(x) \rangle_G$ is a connected graph. A nonsimplicial locally connected vertex is called *eligible*. The graph G'_x with vertex set $V(G'_x) = V(G)$ and edge set $E(G'_x) = E(G) \cup \{xy|\ x, y \in N(x)\}$ is called the *local completion of G at x*. It was shown in [6] that the local completion of a claw-free graph G at x is again claw-free, and if x is eligible, then $c(G'_x) = c(G)$. Based on this result, the following closure concept for claw-free graphs was introduced in [6].

Let G be a claw-free graph and let $\mathrm{cl}^{\mathrm{CF}}(G)$ be a graph obtained from G by recursively performing the local completion operation at eligible vertices, as long as this is possible. The following was proved in [6].

Theorem B [6]. *Let G be a claw-free graph. Then*
(i) $\mathrm{cl}^{\mathrm{CF}}(G)$ *is uniquely determined,*
(ii) $c(\mathrm{cl}^{\mathrm{CF}}(G)) = c(G)$,
(iii) $\mathrm{cl}^{\mathrm{CF}}(G)$ *is the line graph of a triangle-free graph.*

The graph $\mathrm{cl}^{\mathrm{CF}}(G)$ is called the (claw-free) *closure of* G.

Independently, Catlin [4] introduced a reduction technique based on contracting collapsible subgraphs to a vertex. A graph H is *collapsible* if for every even set $S \subset V(H)$ there is a subgraph $\Gamma \subset H$ such that
(i) $H - E(\Gamma)$ is connected,
(ii) $v \in S$ if and only if $d_\Gamma(v)$ is odd.
Every graph has a unique collection of maximal collapsible subgraphs, and contracting each of them to a single vertex affects neither the existence nor the nonexistence of a spanning eulerian subgraph.

Veldman [7] refined the Catlin's technique by handling vertices of degree 1 and 2. This refinement can be described in the following way. For a simple graph H let $D(H) = \{v \in V(H) |\ d_H(v) = 1 \text{ or } 2\}$. For an independent set $X \subset D(H)$, let $I_X(H)$ be the graph obtained from H by contracting one edge incident with each vertex of X. Veldman then defined H as X-collapsible if $I_X(H)$ is collapsible in the Catlin sense.

Both of these reduction techniques are powerful tools for studying hamiltonicity of line graphs. However, their main drawback is that the search for maximal collapsible subgraphs is very difficult.

In this context, a natural question is whether the claw-free closure concept can be strengthened by using line graph techniques or by combining them with closure techniques. (Recall that $\mathrm{cl}^{\mathrm{CF}}(G)$ is a line graph.) A first attempt in this direction was done in paper [3]. We continue in this direction and in the next two sections we show that the reduction techniques of Catlin and Veldman can be reformulated in terms of a closure technique for line graphs. The closure technique is more convenient to use since it avoids the necessity of a search for maximal contractible subgraphs – in many cases even closing (= contracting in $L^{-1}(H)$) a small subgraph can start a “domino effect” resulting in closing the whole graph. We show that the closure is unique and that it strengthens all the above mentioned techniques.

2 A-contractible graphs

If H is a graph and $F \subset H$ is a subgraph of H, then $H|_F$ denotes the graph obtained from H by identifying the vertices of F as a (new) vertex v_F, and by replacing the created loops by pendant edges. Observe that $H|_F$ may contain multiple edges. If H is a graph, $X \subset V(H)$, and $\mathcal{A}$ is a partition of X into subsets, then $E(\mathcal{A})$ denotes the set of all edges a_1a_2 (not necessarily in H) such that a_1, a_2 are in the same element of $\mathcal{A}$. Further $H^{\mathcal{A}}$ denotes the graph with vertex set $V(H^{\mathcal{A}}) = V(H)$ and edge set $E(H^{\mathcal{A}}) = E(H) \cup E(\mathcal{A})$. Note that $E(H)$ and $E(\mathcal{A})$ are considered to be disjoint, i.e., if $e_1 = a_1a_2 \in E(H)$ and $e_2 = a_1a_2 \in E(\mathcal{A})$, then e_1, e_2 are parallel edges in $H^{\mathcal{A}}$.

Let F be a graph and let $A \subset V(F)$. We say that F is A-*contractible*, if for every even subset $X \subset A$ and for every partition $\mathcal{A}$ of X into two-element subsets the graph $F^{\mathcal{A}}$ has a DCT containing all vertices of A and all edges of $E(\mathcal{A})$. Note that this definition allows X to be empty, in which case $F^{\mathcal{A}} = F$. Also, if F is A-contractible, then F is A'-contractible for any $A' \subset A$ (since every subset X of A' is a subset of A).

Theorem 1. *Let F be a connected graph and let $A \subset V(F)$. Then F is A-contractible if and only if*

$$d_T(H) = d_T(H|_F)$$

for every graph H such that $F \subset H$ and $A_H(F) = A$.

Proof. Obviously, if $F \subset H$, every closed trail T in H has a corresponding closed trail in $H|_F$, the closed trail $T|_F$, dominating at least as many edges as T. This immediately implies that $d_T(H) \le d_T(H|_F)$.

First suppose that F is A-contractible, and let T' be a maximum closed trail in $H|_F$. If T' does not contain v_F, then T' is also a closed trail in H, implying $d_T(H|_F) \le d_T(H)$, as desired. Hence suppose T' contains v_F. The edges of T' which are in H determine a system of trails $\mathcal{P} = \{P_1, \dots, P_k\}$ such that every endvertex of every $P_i \in \mathcal{P}$ is a vertex of attachment of F, and every closed trail in $\mathcal{P}$ contains at least one vertex of attachment of F. Choose the trails P_i such that $|\mathcal{P}|$ is minimum. Then every $x \in A_H(F)$ is an endvertex of at most one trail from $\mathcal{P}$ (otherwise we can reduce $|\mathcal{P}|$ by joining some of the trails ending at x).

Set $X = \{x \in A_H(F)|\ x$ is an endvertex of some $P_i \in \mathcal{P}\}$ and $\mathcal{A} = \{A_1, \dots, A_k\}$, where A_i is the (two-element) set of endvertices of P_i, $i = 1, \dots, k$. By the assumption, F is A-contractible and hence $F^{\mathcal{A}}$ has a DCT Q. The trail Q determines in F a system of trails $Q_1, \dots, Q_k$ such that every Q_i has its two endvertices in two different elements of $\mathcal{A}$. The trails Q_i together with the system $\mathcal{P}$ then form a closed trail in H, dominating at least as many edges as T'. Hence $d_T(H|_F) \le d_T(H)$, implying $d_T(H|_F) = d_T(H)$.

Suppose F is not A-contractible. Then there is an $X \subset A$ and a partition $\mathcal{A}$ of X into two-element sets such that $F^{\mathcal{A}}$ has no DCT containing all vertices of A and all edges of $E(\mathcal{A})$. Let $\mathcal{A} = \{\{x_1', x_1''\}, \dots, \{x_k', x_k''\}\}$, and construct a graph H by joining k vertex disjoint x_i', x_i''-paths P_i of length at least 3, $i = 1, \dots, k$, to F and by attaching a pendant edge to every vertex of F in $A \setminus X$. If T is a DCT in H, then T contains all the paths $P_1, \dots, P_k$, implying that $E(T) \cap E(F)$ determines a DCT in $F^{\mathcal{A}}$, containing all vertices of A and all edges of $E(\mathcal{A})$, a contradiction. Hence H has no DCT. Since clearly $H|_F$ has a DCT, we have $d_T(H) < d_T(H|_F)$. ∎

Our next theorem shows that a contractible graph remains contractible after a partial contraction. For any two sets $A, B \subset V(F)$, $A \cap B \neq \emptyset$, $A|_B$ denotes the set $(A \setminus B) \cup \{v_B\} \subset V(F|_B)$. For $A \cap B = \emptyset$ we set $A|_B = A$.

Theorem 2. *Let F be a graph and let $A, B \subset V(F)$. If F is A-contractible, then $F|_B$ is $A|_B$-contractible.*

Equivalently, Theorem 2 says that the family $\mathcal{F} = \{(F, A)|\ F \text{ is } A\text{-contractible}\}$ is closed under partial contraction.

Proof. Let $X \subset A|_B$ be an even subset and let $\mathcal{A}$ be a partition of X into two-element subsets. If $v_B \notin X$, then set $X' = X$, otherwise choose arbitrarily a $b \in A \cap B$, set $X' = (X \setminus \{v_B\}) \cup \{b\}$ and let $\mathcal{A}'$ be the partition of X' obtained by replacing v_B by b. Then $X' \subset A$ is even and $\mathcal{A}'$ is a partition of X' into two-element subsets. By the A-contractibility of F, there is a DCT T' in F containing all vertices of A' and all edges of $E(\mathcal{A})$. But then the trail $T = T'|_B$ is a DCT in $F|_B$ containing all vertices of $A|_B$ and all edges of $E(\mathcal{A})$. Hence $F|_B$ is $A|_B$-contractible. ∎

Theorem 2 has the following consequence, showing that contractible graphs can be "built" from smaller ones.

Corollary 3. *Let F be a graph, $A \subset V(F)$, and let F_1, F_2 be subgraphs of F such that $F = F_1 \cup F_2$ and $A_F(F_1 \cap F_2) \subset A$. Let $A_i = V(F_i) \cap A$, $i = 1, 2$. If F_i is A_i-contractible, $i = 1, 2$, then F is A-contractible.*

Proof. Let H be an arbitrary graph such that $F_1, F_2 \subset H$ and $A_H(F_i) = A_i$, $i = 1, 2$. Set $H' = H|_{F_1}$ and $F_2' = F_2|_{F_1}$. Since F_1 is A_1-contractible, $d_T(H) = d_T(H')$. By Theorem 2, F_2' is $A_2|_{V(F_1)}$-contractible and hence further $d_T(H') = d_T(H'|_{F_2'})$. Obviously $H'|_{F_2'} = H|_{F_1 \cup F_2}$, implying $d_T(H) = d_T(H|_{F_1 \cup F_2})$. Since H is arbitrary, the graph $F_1 \cup F_2$ is $A_1 \cup A_2$-contractible by Theorem 1. ∎

In the next section we show that the concept of A-contractibility in H can be reformulated as a closure concept in $L(H)$.

3 Closure concept

If $F \subset H$, then the set of all edges of H, having at least one vertex in $V(F)$, corresponds in $H|_F$ to the set of all edges that contain the vertex v_F. In the line graph this means that the set of all vertices, corresponding to the set of edges of H with at least one vertex in $V(F)$, induces a clique in $L(H|_F)$. Equivalently, $L(H|_F)$ is obtained from $L(H)$ by making the neighborhood of the graph $L(F)$ complete.

Following [6], we introduce similar terminology. Let G be a graph and M an induced subgraph of G. The graph G'_M with vertex set $V(G'_M) = V(G)$ and edge set $E(G'_M) = E(G) \cup \{xy| \; x, y \in N(M)\}$ is called the *local completion of* G *at* M. Obviously, if $F \subset H$, $G = L(H)$ and $M = L(F)$, then M is an induced subgraph of G, $L(H|_F) = G'_M$ and $d_T(H) = c(G)$. We say that the induced subgraph M of G is *eligible*, if $F = L^{-1}(M)$ is $A_H(F)$-contractible. If we speak of eligibility of some line graph M without pointing out explicitly any of its supergraphs G, we always suppose that in $F = L^{-1}(M)$ a subset $A \subset V(F)$ is specified. (Note that this is equivalent to specifying a collection of cliques of attachment in M.)

In this terminology, Theorem 1 has the following immediate consequence for line graphs.

Corollary 4. *Let G be a line graph and let $M \subset G$ be eligible. Then*

$$c(G) = c(G'_M).$$

It should be noted that if M is not eligible, then M is an induced subgraph of some graph $\tilde{G}$ such that $\tilde{G}$ is a line graph and $c(\tilde{G}) < c(\tilde{G}'_M)$.

The local completion of G at M consists in adding edges to G such that $\langle M \rangle_{G'_M}$ is complete. Theorem 1 and Corollary 4 show that the completion of M leaves the circumference of G unchanged for any supergraph G if and only if M is eligible. Thus, the definition of eligibility as given in this paper is the most general one, and no further generalization of this type of closure is possible.

Examples. 1. Catlin [4] introduced a concept of collapsible graphs. Since every vertex of $H|_F$, for which $|\theta^{-1}(x)| > 1$ (in the notation of [4]), is incident to at least one pendant edge, from Theorem 8(vii) of [4] and from Theorem A it follows that H has a DCT if and only if $H|_F$ has a DCT. If H has no DCT, then we can without loss of generality restrict ourselves to the subgraph given by the edges of its maximum closed trail. By Theorem 1, we have the following.

(i) Every collapsible graph F is $V(F)$-contractible.

(ii) If G is a line graph and $M \subset G$ is an induced subgraph such that $L^{-1}(M)$ is collapsible, then M is eligible.

2. Veldman [7] refined the Catlin's reduction technique by introducing a concept of X-collapsibility of a graph F (where X is an independent subset of the set $D(F)$ of vertices of F of degree 1 or 2). From Lemma 5 of [7] we obtain, similarly as in the previous example, the following statements.

(i) If F is X-collapsible for some independent set $X \subset D(F)$, then F is $(V(F) \setminus D(F))$-contractible.

(ii) If G is a line graph and $M \subset G$ is an induced subgraph such that $L^{-1}(M)$ is X-collapsible, then M is eligible.

3. A closure concept in the class of claw-free graphs based on the operation of a local completion at eligible vertices was introduced in [6] by the first author. It is easy to see that in the special case of a line graph, a nonsimplicial vertex x of $L(H)$ is locally connected if and only if the corresponding edge $e = u_1u_2$ of H is in a triangle (say, $T = \langle\{u_1, u_2, v\}\rangle_H$), or in a cycle of length 2. In the first case, let y_1, y_2 be the vertices of $L(H)$ corresponding to the edges u_1v, u_2v. By [4], the triangle is collapsible, and it is apparent that $L(H|_T)$ is obtained from $L(H)$ by local completion at x if $v \notin A_H(F)$, or by two local completions at x and at one of y_1, y_2, if $v \in A_H(F)$. The second case (e is in a cycle of length 2) is similar.

4. In [3], the closure for claw-free graphs was strengthened by showing that $c(G'_M) = c(G)$ if $G = L(H)$ is a line graph, H is triangle-free, and $F = L^{-1}(M)$ is a C_4, C_5 or C_6 such that $|A_H(F)| = 3$ and the vertices of $Y = V(F) \setminus A_H(F)$ (of degree 2 in H) are not consecutive on F. In the terminology of this paper, let F_1, F_2, F_3 be the graphs of Figure 1 and $A_i = \{a_1, a_2, a_2\} \subset V(F_i)$ (the vertices of attachment $a_i \in A_i$ are double-circled in the figure).

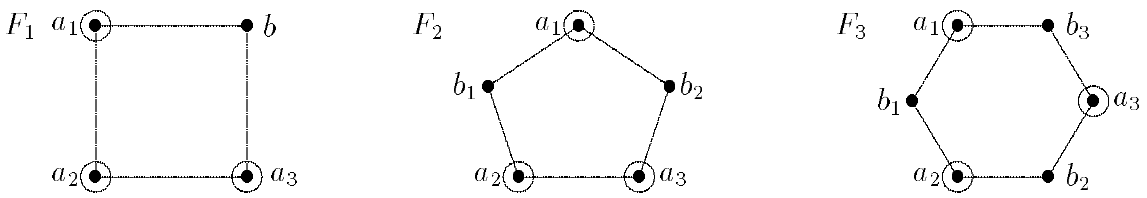

Figure 1

Then it is straightforward to verify that each of these graphs is A-contractible. We check (up to a symmetry) all possible even subsets $X \subset A$ and their partitions $\mathcal{A}$ (in fact, there is only one since $|\mathcal{A}| = 3$), and we show the corresponding DCT T in $F^{\mathcal{A}}$. The edge of $E(\mathcal{A})$ is overlined.

	X	$\mathcal{A}$	T
Graph F_1:	$\emptyset$	$\{\emptyset\}$	$a_1a_2a_3ba_1$
	$\{a_1, a_2\}$	$\{\{a_1, a_2\}\}$	$\overline{a_1a_2}a_3ba_1$
	$\{a_1, a_3\}$	$\{\{a_1, a_3\}\}$	$\overline{a_1a_3}a_2a_1$
Graph F_2:	$\emptyset$	$\{\emptyset\}$	$a_1b_1a_2a_3b_2a_1$
	$\{a_1, a_2\}$	$\{\{a_1, a_2\}\}$	$\overline{a_1a_2}a_3b_2a_1$
	$\{a_2, a_3\}$	$\{\{a_2, a_3\}\}$	$\overline{a_2a_3}b_2a_1b_1a_2$
Graph F_3:	$\emptyset$	$\{\emptyset\}$	$a_1b_1a_2b_2a_3b_3a_1$
	$\{a_1, a_2\}$	$\{\{a_1, a_2\}\}$	$\overline{a_1a_2}b_2a_3b_3a_1$

5. Let F be the graph of Figure 2 and let $A = \{a_1, a_2, a_3, a_4\} \subset V(F)$ (the vertices of attachment $a_i \in A$ are again double-circled in the figure). Then F is A-contractible, but neither of the techniques of Examples 1 – 4 applies to F.

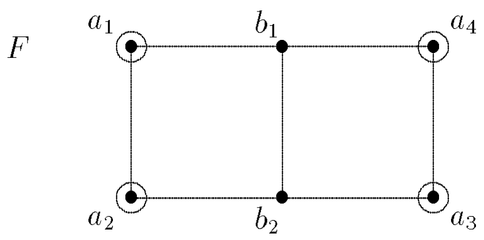

Figure 2

Indeed, F is not collapsible (in the sense of [4]) since for the set $S = V(F)$ there is no S-subgraph (a subgraph Γ such that $G - E(\Gamma)$ is connected and $v \in S$ if and only if $d_\Gamma(v)$ is odd). Also, F cannot be an X-collapsible subgraph of any graph H in the sense of [7], since it is not collapsible and all its vertices of degree 2 are its vertices of attachment in H. Similarly, the techniques of Examples 3 and 4 are not applicable as well. However, it is not difficult to check that F is A-contractible.

X	$\mathcal{A}$	T
$\emptyset$	$\{\emptyset\}$	$a_1a_2b_2a_3a_4b_1a_1$
$\{a_1, a_2\}$	$\{\{a_1, a_2\}\}$	$\overline{a_1a_2}b_2a_3a_4b_1a_1$
$\{a_1, a_3\}$	$\{\{a_1, a_3\}\}$	$\overline{a_1a_3}a_4b_1b_2a_2a_1$
$\{a_1, a_4\}$	$\{\{a_1, a_4\}\}$	$\overline{a_1a_4}a_3b_2a_2a_1$
A	$\{\{a_1, a_2\}, \{a_3, a_4\}\}$	$\overline{a_1a_2}b_2\overline{a_3a_4}b_1a_1$
A	$\{\{a_1, a_3\}, \{a_2, a_4\}\}$	$\overline{a_1a_3}b_2\overline{a_2a_4}b_1a_1$
A	$\{\{a_1, a_4\}, \{a_2, a_3\}\}$	$\overline{a_1a_4}\ \overline{a_3a_2}b_2b_1a_1$

Our next result shows that, in a line graph G, performing a local completion at some eligible subgraph does not affect elibigibility of other eligible subgraphs of G.

Theorem 5. *Let G_1, G_2 be two eligible subgraphs of a line graph $G = L(H)$ and let $G' = G'_{G_1}$ be the local completion of G at G_1. Then the subgraph $\langle V(G_2)\rangle_{G'}$ is eligible in G'.*

Proof. Let $F_i = L^{-1}(G_i)$, $i = 1, 2$. If $V(F_1) \cap V(F_2) = \emptyset$, there is nothing to prove. Hence let $V(F_1) \cap V(F_2) = M \neq \emptyset$. Then $\langle V(G_2)\rangle_{G'}$ corresponds to $F_2|_M$, which is $A_{H|_{F_1}}(F_2|_M)$-contractible by Theorem 2. ∎

Having the results established in Theorems 2 and 5, we can introduce the main concept of this paper. Although a characterization of eligible graphs is given, we are not able to give a complete list of them (and we doubt this might ever be possible). In practical situations, the closure concept will always be used with a certain restricted family of "known" eligible graphs. This is the main motivation of the following definition.

Let G be a line graph and let $\mathcal{C}$ be a family of eligible subgraphs of G. We say that the family $\mathcal{C}$ is *complete*, if the corresponding family $\mathcal{F} = \{(F, A) |\ L(F) \in \mathcal{C}, F$ is A-contractible$\}$ is closed under partial contraction. (Specifically, by Theorems 2 and 5, this is always the case if $\mathcal{C}$ is the family of all eligible graphs).

The *$\mathcal{C}$-closure of G* is a graph $\mathrm{cl}^{\mathcal{C}}(G)$ for which there is a sequence of graphs $G_1, \ldots, G_t$ such that

(i) $G_1 = G$, $G_t = \mathrm{cl}^{\mathcal{C}}(G)$,

(ii) $G_{i+1} = (G_i)'_M$ for some eligible induced subgraph $M \subset G_i$, $M \in \mathcal{C}$,

(iii) $G_t = \mathrm{cl}^{\mathcal{C}}(G)$ contains no induced subgraph from $\mathcal{C}$.

Theorem 6. *Let G be a line graph and let $\mathcal{C}$ be a complete family of eligible line graphs. Then*

(i) $\mathrm{cl}^{\mathcal{C}}(G)$ *is uniquely determined,*

(ii) $c(G) = c(\mathrm{cl}^{\mathcal{C}}(G))$.

Proof. (i) Let G', G'' be two $\mathcal{C}$-closures of G and suppose that $E(G') \setminus E(G'') \neq \emptyset$. Let $G = G'_1, \ldots, G'_s = G'$ and $G = G''_1, \ldots, G''_t = G''$ be the corresponding sequences of local completions. Let j be the smallest integer for which there is an edge $e = xy \in E(G'_j) \setminus E(G'')$. Then $x, y \in V(M)$ for some eligible subgraph $M \subset G'_{j-1}$. But $E(G'_{j-1}) \subset E(G'')$, implying $M \subset G''$. By Theorem 5 and since $\mathcal{C}$ is complete, $\langle V(M) \rangle_{G''}$ is eligible, implying $e \in E(G'')$, a contradiction.

(ii) From Corollary 4, $c(G) = c(\mathrm{cl}^{\mathcal{C}}(G))$. ■

There are two observations to be emphasized. The first one is that the concept of $\mathcal{C}$-closure can be extended to the class of claw-free graphs. To see this, let G be a claw-free graph (not necessarily a line graph) and let $\mathrm{cl}^{\mathrm{CF}}(G)$ be the closure of G introduced in [6]. Then the concept of $\mathcal{C}$-closure is extended to a closure defined on the class of claw-free graphs by setting

$$\mathrm{cl}(G) = \mathrm{cl}^{\mathcal{C}}(\mathrm{cl}^{\mathrm{CF}}(G)).$$

Since both the closures are unique and preserve the value of circumference, the same holds for $\mathrm{cl}(G)$.

Secondly, if $\mathcal{C}$ contains all collapsible and X-collapsible graphs, then $\mathrm{cl}(G)$ is the line graph of a graph which is reduced both in the sense of Catlin [4] and Veldman [7]. Moreover, Example 5 shows that $\mathrm{cl}(G)$ is stronger than the techniques of [4] and [7].

Examples. 6. Each of the graphs F_i in Figure 3 is A_i-contractible (the vertices of attachment $a_i \in A_i$ are double-circled). We verify the contractibility of the graphs $F_1, \ldots, F_6$ in a way similar to that of Examples 4 and 5.

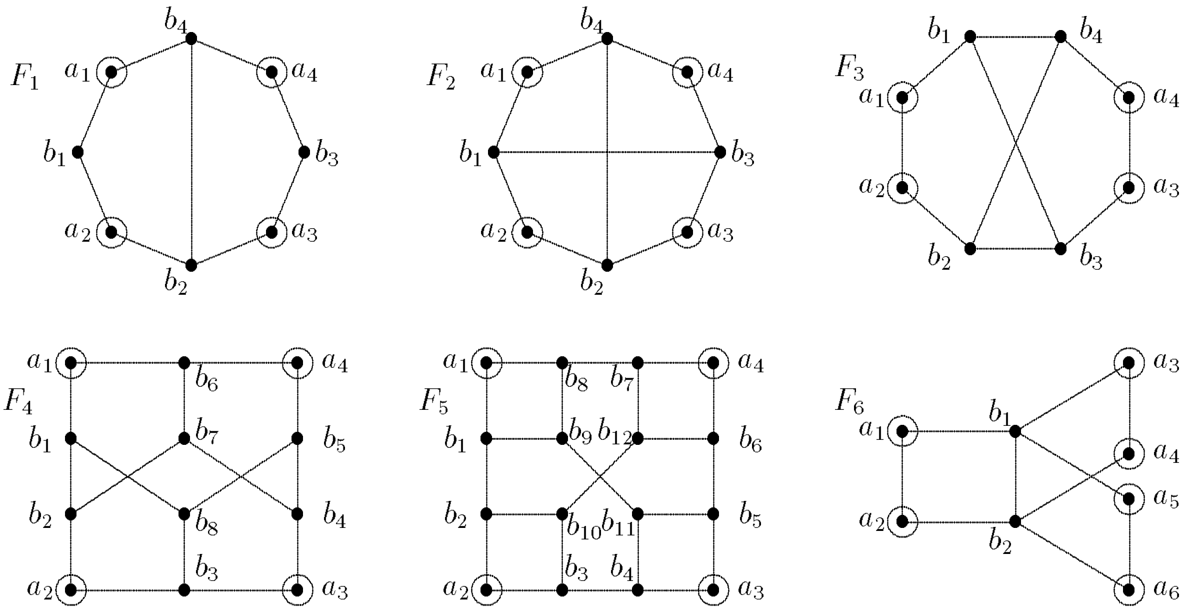

Figure 3

Graph F_1	X	$\mathcal{A}$	T
	$\emptyset$	$\{\emptyset\}$	$a_1b_1a_2b_2a_3b_3a_4b_4a_1$
	$\{a_1,a_2\}$	$\{\{a_1,a_2\}\}$	$\overline{a_1a_2}b_2a_3b_3a_4b_4a_1$
	$\{a_1,a_3\}$	$\{\{a_1,a_3\}\}$	$\overline{a_1a_3}b_3a_4b_4b_2a_2b_1a_1$
	$\{a_1,a_4\}$	$\{\{a_1,a_4\}\}$	$\overline{a_1a_4}b_3a_3b_2a_2b_1a_1$
	A	$\{\{a_1,a_2\},\{a_3,a_4\}\}$	$\overline{a_1a_2}b_2\overline{a_3a_4}b_4a_1$
	A	$\{\{a_1,a_3\},\{a_2,a_4\}\}$	$\overline{a_1a_3}b_2\overline{a_2a_4}b_4a_1$
	A	$\{\{a_1,a_4\},\{a_2,a_3\}\}$	$\overline{a_1a_4}b_4b_2\overline{a_3a_2}b_1a_1$
Graph F_2	X	$\mathcal{A}$	T
	$\emptyset$	$\{\emptyset\}$	$a_1b_1a_2b_2a_3b_3a_4b_4a_1$
	$\{a_1,a_2\}$	$\{\{a_1,a_2\}\}$	$\overline{a_1a_2}b_2a_3b_3a_4b_4a_1$
	$\{a_1,a_3\}$	$\{\{a_1,a_3\}\}$	$\overline{a_1a_3}b_3a_4b_4b_2a_2b_1a_1$
	A	$\{\{a_1,a_2\},\{a_3,a_4\}\}$	$\overline{a_1a_2}b_2\overline{a_3a_4}b_3b_1a_1$
	A	$\{\{a_1,a_3\},\{a_2,a_4\}\}$	$\overline{a_1a_3}b_3\overline{a_4a_2}b_2b_4a_1$
Graph F_3	X	$\mathcal{A}$	T
	$\emptyset$	$\{\emptyset\}$	$a_1a_2b_2b_3a_3a_4b_4b_1a_1$
	$\{a_1,a_2\}$	$\{\{a_1,a_2\}\}$	$\overline{a_1a_2}b_2b_3a_3a_4b_4b_1a_1$
	$\{a_1,a_3\}$	$\{\{a_1,a_3\}\}$	$\overline{a_1a_3}a_4b_4b_1b_3b_2a_2a_1$
	$\{a_1,a_4\}$	$\{\{a_1,a_4\}\}$	$\overline{a_1a_4}a_3b_3b_1b_4b_2a_2a_1$
	A	$\{\{a_1,a_2\},\{a_3,a_4\}\}$	$\overline{a_1a_2}b_2b_3\overline{a_3a_4}b_4b_1a_1$
	A	$\{\{a_1,a_3\},\{a_2,a_4\}\}$	$\overline{a_1a_3}b_3b_2\overline{a_2a_4}b_4b_1a_1$
	A	$\{\{a_1,a_4\},\{a_2,a_3\}\}$	$\overline{a_1a_4}b_4b_2\overline{a_2a_3}b_3b_1a_1$

Graph F_4	X	$\mathcal{A}$	T
	$\emptyset$	$\{\emptyset\}$	$a_1b_1b_2a_2b_3a_3b_4b_5a_4b_6a_1$
	$\{a_1,a_2\}$	$\{\{a_1,a_2\}\}$	$\overline{a_1a_2}b_3a_3b_4b_7b_6a_4b_5b_8b_1a_1$
	$\{a_1,a_3\}$	$\{\{a_1,a_3\}\}$	$\overline{a_1a_3}b_3a_2b_2b_7b_4b_5a_4b_6a_1$
	$\{a_1,a_4\}$	$\{\{a_1,a_4\}\}$	$\overline{a_1a_4}b_6b_7b_2a_2b_3a_3b_4b_5b_8b_1a_1$
	A	$\{\{a_1,a_2\},\{a_3,a_4\}\}$	$\overline{a_1a_2}b_2b_7b_4\overline{a_3a_4}b_5b_8b_1a_1$
	A	$\{\{a_1,a_3\},\{a_2,a_4\}\}$	$\overline{a_1a_3}b_4b_7b_2\overline{a_2a_4}b_5b_8b_1a_1$
	A	$\{\{a_1,a_4\},\{a_2,a_3\}\}$	$\overline{a_1a_4}b_5b_8b_3\overline{a_2a_3}b_2b_7b_6a_1$

Graph F_5	X	$\mathcal{A}$	T
	$\emptyset$	$\{\emptyset\}$	$a_1b_1b_9b_{11}b_5a_3b_4b_3a_2b_2b_{10}b_{12}b_6a_4b_7b_8a_1$
	$\{a_1,a_2\}$	$\{\{a_1,a_2\}\}$	$\overline{a_1a_2}b_2b_{10}b_3b_4a_3b_5b_6a_4b_7b_8b_9b_1a_1$
	$\{a_1,a_3\}$	$\{\{a_1,a_3\}\}$	$\overline{a_1a_3}b_4b_{11}b_5b_6a_4b_7b_{12}b_{10}b_3a_2b_2b_1b_9b_8a_1$
	A	$\{\{a_1,a_2\},\{a_3,a_4\}\}$	$\overline{a_1a_2}b_2b_{10}b_3b_4b_{11}b_5\overline{a_3a_4}b_6b_{12}b_7b_8b_9b_1a_1$
	A	$\{\{a_1,a_3\},\{a_2,a_4\}\}$	$\overline{a_1a_3}b_4b_{11}b_5b_6b_{12}b_7\overline{a_4a_2}b_3b_{10}b_2b_1b_9b_8a_1$

Graph F_6	X	$\mathcal{A}$	T
	$\emptyset$	$\{\emptyset\}$	$a_1a_2b_2b_1a_3a_4b_2a_6a_5b_1a_1$
	$\{a_1,a_2\}$	$\{\{a_1,a_2\}\}$	$\overline{a_1a_2}b_2a_6a_5b_1a_3a_4b_2b_1a_1$
	$\{a_1,a_3\}$	$\{\{a_1,a_3\}\}$	$\overline{a_1a_3}a_4b_2b_1a_5a_6b_2a_2a_1$
	$\{a_1,a_4\}$	$\{\{a_1,a_4\}\}$	$\overline{a_1a_4}a_3b_1a_5a_6b_2a_2a_1$
	$\{a_1,a_2,a_3,a_4\}$	$\{\{a_1,a_2\},\{a_3,a_4\}\}$	$\overline{a_1a_2}b_2a_6a_5b_1\overline{a_3a_4}b_2b_1a_1$
	$\{a_1,a_2,a_3,a_4\}$	$\{\{a_1,a_3\},\{a_2,a_4\}\}$	$\overline{a_1a_3}b_1a_5a_6b_2\overline{a_4a_2}a_1$
	$\{a_1,a_2,a_3,a_4\}$	$\{\{a_1,a_4\},\{a_2,a_3\}\}$	$\overline{a_1a_4}b_2a_6a_5b_1\overline{a_3a_2}a_1$
	$\{a_1,a_2,a_3,a_5\}$	$\{\{a_1,a_2\},\{a_3,a_5\}\}$	$\overline{a_1a_2}b_2a_6\overline{a_5a_3}a_4b_2b_1a_1$
	$\{a_1,a_2,a_3,a_5\}$	$\{\{a_1,a_3\},\{a_2,a_5\}\}$	$\overline{a_1a_3}a_4b_2a_6\overline{a_5a_2}a_1$
	$\{a_1,a_2,a_3,a_6\}$	$\{\{a_1,a_2\},\{a_3,a_6\}\}$	$\overline{a_1a_2}b_2a_4\overline{a_3a_6}a_5b_1a_1$
	$\{a_1,a_2,a_3,a_6\}$	$\{\{a_1,a_3\},\{a_2,a_6\}\}$	$\overline{a_1a_3}a_4b_2\overline{a_2a_6}a_5b_1a_1$
	$\{a_1,a_2,a_3,a_6\}$	$\{\{a_1,a_6\},\{a_2,a_3\}\}$	$\overline{a_1a_6}a_5b_1b_2a_4\overline{a_3a_2}a_1$
	A	$\{\{a_1,a_2\},\{a_3,a_4\},\{a_5,a_6\}\}$	$\overline{a_1a_2}b_2\overline{a_6a_5}b_1b_2\overline{a_4a_3}b_1a_1$
	A	$\{\{a_1,a_2\},\{a_3,a_5\},\{a_4,a_6\}\}$	$\overline{a_1a_2}b_2\overline{a_4a_6}\ \overline{a_5a_3}b_1a_1$
	A	$\{\{a_1,a_2\},\{a_3,a_6\},\{a_4,a_5\}\}$	$\overline{a_1a_2}b_2\overline{a_4a_5}\ \overline{a_6a_3}b_1a_1$
	A	$\{\{a_1,a_3\},\{a_2,a_5\},\{a_4,a_6\}\}$	$\overline{a_1a_3}\ \overline{a_4a_6}\ \overline{a_5a_2}b_2b_1a_1$
	A	$\{\{a_1,a_3\},\{a_2,a_6\},\{a_4,a_5\}\}$	$\overline{a_1a_3}\ \overline{a_4a_5}\ \overline{a_6a_2}b_2b_1a_1$
	A	$\{\{a_1,a_4\},\{a_3,a_6\},\{a_2,a_5\}\}$	$\overline{a_1a_4}\ \overline{a_3a_6}\ \overline{a_5a_2}b_2b_1a_1$

Note that none of the graphs $F_1,\dots,F_6$ is collapsible in the sense of [4] or [7]. It can be checked that F_3 is A-contractible even for $A=\{a_1,a_2,a_3,a_4,b_1,b_4\}$. We omit the (tedious) proof of this fact.

7. Let H be the Petersen graph. Then H contains a subgraph F isomorphic to the graph F_2 from the previous example with $A_H(F)=\{a_1,a_2,a_3,a_4\}$ (see Figure 4, vertices of F are labeled as in Fig. 3). In $H|_F$, every edge is in a cycle of length 2 or 3. Hence the $\mathcal{C}$-closure of $L(H)$ is a complete graph. However, it is not difficult to check H contains no collapsible subgraph.

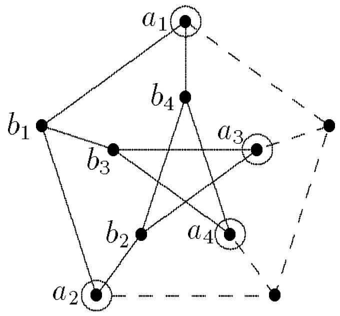

Figure 4

This example shows that there is often no need to search for maximal contractible subgraphs since contracting (or, in $L(H)$, completing) even a small subgraph can create triangles, and the subsequent "domino effect" results in turning the whole $L(H)$ into a clique.

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References

[1] Beineke, L.W.: Characterizations of derived graphs. J. Combin. Theory Ser. B 9(1970) 129-135

[2] Bondy, J.A.; Murty, U.S.R.: Graph theory with applications. Macmillan, London and Elsevier, New York, 1976.

[3] Broersma, H.J.; Ryjáček, Z.: Strengthening the closure concept in claw-free graphs. Discrete Math. (to appear).

[4] Catlin, P.A.: A reduction technique to find spanning eulerian subgraphs. J. Graph Theory 12 (1988), 29-44.

[5] F. Harary, C. St.J.A. Nash-Williams: On eulerian and hamiltonian graphs and line graphs. Canad. Math. Bull. 8 (1965), 701-709.

[6] Ryjáček, Z.: On a closure concept in claw-free graphs. J. Combin. Theory Ser. B 70(1997), 217-224.

[7] Veldman, H.J.: On dominating and spanning circuits in graphs. Discrete Math. 124 (1994), 229-239