Regular Clique Covers of Graphs

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Abstract

A family of cliques in a graph G is said to be *p-regular* if any two cliques in the family intersect in exactly p vertices. A graph G is said to have a *p-regular k-clique cover* if there is a p-regular family $\mathcal{H}$ of k-cliques of G such that each edge of G belongs to a clique in $\mathcal{H}$. Such a p-regular k-clique cover is *separable* if the complete subgraphs of order p that arise as intersections of pairs of distinct cliques of $\mathcal{H}$ are mutually vertex-disjoint.

For any given integers p, k, ℓ; $p < k$, we present bounds on the smallest order of a graph that has a p-regular k-clique cover with exactly ℓ cliques, and we describe all graphs that have p-regular separable k-clique covers with ℓ cliques.

1 Introduction

An orthogonal double cover of a complete graph K_n by a graph H is a collection $\mathcal{H}$ of spanning subgraphs of K_n, all isomorphic to H, such that each edge of K_n is contained in exactly two subgraphs in $\mathcal{H}$ and any two distinct subgraphs in $\mathcal{H}$ share exactly one edge. This concept was first studied by Demetrovics, Füredi and Katona [5] and it has since received considerable attention [10]. As for general graphs H the problem of existence of an orthogonal double cover is considered to be decidedly hard; results have mainly been obtained for special graphs such as trees [9, 11, 16, 19, 22, 24, 25], triangle factors [1, 7], short cycles [9, 23], graphs of maximum degree two [4, 13], and almost-hamiltonian cycles [19, 20, 21]. An important special case occurs when H is a complete graph K_k, in which case the existence of an orthogonal double cover of K_n by H is equivalent to the existence of a symmetric $2-(n, k, 2)$ design. Only a small number of such designs are known (c.f. [3] pp. 14–16 and 75–81).

More recent papers in the area have introduced several generalizations. Relaxing the first condition by requiring that any two copies of H share at most one edge gives a suborthogonal double cover of K_n by H; such covers (for $H = P_4$) and their asymptotic existence (for arbitrary H) were studied in [17]. A further variation is the concept of a generalized graph design with parameters $(n, H, \lambda; F)$, which is a family of isomorphic copies of H in K_n such that each edge of K_n is contained in exactly λ copies of H and any two distinct copies of H intersect in a subgraph isomorphic to F. The case when H is a friendship graph (a set of triangles joined at a single vertex) was completely solved in [12] and [6]. Generalized graph designs with doubly transitive automorphism groups were investigated in [2].

As one can see, there is a fair amount of research devoted to covers of *complete* graphs by various types of more or less *general* graphs. Our point of departure is to interchange the roles of the two classes of graphs and ask about covers of *general* graphs by *cliques*, that is, complete subgraphs that are maximal with respect to containment. Emphasis will be given on covers that have certain "design-like" regularity features.

The following are the central concepts of our paper. A collection of cliques in a graph G is *p-regular* if any two cliques in the family intersect in exactly p vertices. We will say that a graph G has a *p-regular k-clique cover* if there is a p-regular family $\mathcal{H}$ of cliques of order k in G such that each edge of G belongs to a clique in $\mathcal{H}$. Furthermore, such a p-regular k-clique cover of G will be called *separable* if the subgraphs isomorphic to K_p obtained as intersections of pairs of distinct cliques in $\mathcal{H}$ are pairwise vertex-disjoint (and, consequently, each edge of G belongs to exactly one or two cliques in $\mathcal{H}$).

The separability condition is very strong. In Section 2 we will characterize all triples p, k, ℓ for which there exists a graph G that admits a separable p-regular k-clique cover with exactly ℓ cliques, and we will also show that such a graph is uniquely determined by p, k and ℓ. A graph that has a separable p-regular k-clique cover with exactly ℓ cliques may have several such covers for a given triple p, k, ℓ, and we will see for which triples such a cover is unique.

Without the separability assumption there is more flexibility, and examples of graphs that have a p-regular k-clique covers are abundant. The problem we address is to determine, for a given triple of integers p, k, ℓ; $p < k$, the smallest order $\mu(p, k, \ell)$ of a graph that admits a p-regular k-clique cover with exactly ℓ cliques. Equivalently, but on a less formal level, for a family of ℓ distinct copies of the complete graph K_k we ask about the smallest order of a graph that can be constructed by pairwise "gluing" the copies "along" complete subgraphs of order p in such a way that the ℓ complete graphs will be k-cliques in the new graph. We present a variety of bounds on $\mu(p, k, \ell)$ and show that in some cases the bounds are best possible. We show that the problem of determining $\mu(p, k, \ell)$ is equivalent to an interesting problem in design theory.

2 Separable clique covers

We begin with studying the triples p, k, ℓ for which there are graphs that have a separable p-regular k-clique cover with ℓ cliques. We discuss small values of ℓ first. For $\ell = 1$, K_k is the unique graph with a p-regular k-clique cover consisting of a single clique. If $\ell = 2$, then since $p < k$, the unique graph G that has a separable p-regular k-clique cover with two k-cliques is formed by two copies of K_k intersecting in exactly p vertices. Going one step further, if $\ell = 3$ then $k \geq 2p$. In the case when $k > 2p$, we again have a unique graph G with a separable p-regular k-clique cover with three k-cliques. For $k = 2p$, the only way to "glue" three copies of K_{2p} in such a way that they pairwise intersect in disjoint complete subgraphs of order p results in a K_{3p}, but then the three subgraphs isomorphic to K_{2p} are not cliques.

Generalizing these observations we now characterize all triples p, k, ℓ with

$\ell \geq 4$ for which there exists a graph G that admits a separable p-regular k-clique cover with exactly ℓ cliques, and establish the uniqueness of such a graph.

Theorem 1. *Let p, k, ℓ be positive integers, $\ell \geq 4$. Then there exists a graph G that admits a separable p-regular k-clique cover with ℓ cliques of order k if and only if $k \geq (\ell-1)p$. Moreover, for such a triple p, k, ℓ the corresponding graph G is unique, and it has a unique separable p-regular k-clique cover consisting of ℓ cliques if and only if $(k, \ell) \neq (3p, 4)$.*

Proof. Let G be a graph and let $\mathcal{H} = \{H_1, H_2, \ldots, H_\ell\}$ be a family of k-cliques of G. If $\mathcal{H}$ forms a separable p-regular cover of G, then for $2 \leq i \leq \ell$ the sets $V(H_1) \cap V(H_i)$ are pairwise disjoint and contain p vertices each. Therefore, $k = |V(H_1)| \geq (\ell-1)p$.

Conversely, let $\ell \geq 4$ and $k \geq (\ell-1)p$. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_\ell\}$ be a collection of ℓ disjoint copies of the graph K_k. For $1 \leq i \leq \ell$ let $V_{i1}, V_{i2}, \ldots, V_{i\ell}$ be a partition of the vertex set $V(H_i)$ such that $|V_{ij}| = p$ for every $i \neq j$ (or if $k = \ell p$); it follows that $|V_{ii}| = k - (\ell-1)p \geq 0$. We now identify, for each i and j such that $1 \leq i < j \leq \ell$, the two vertex sets V_{ij} and V_{ji}. In the graph G arising this way, the family $\mathcal{H}$ clearly forms a separable p-regular k-clique cover with ℓ cliques. On the other hand, let G' be any graph that admits, for the given parameters p, k, ℓ, a separable p-regular k-clique cover $\mathcal{H}' = \{H_1', H_2', \ldots, H_\ell'\}$. Then we may establish an isomorphism from G' onto G by mapping, for $1 \leq i \leq \ell$, the vertex sets $V(H_i')$ onto the sets $V(H_i)$ in such a way that the intersections $V(H_i') \cap V(H_j')$ are mapped onto the sets obtained by identifying V_{ij} with V_{ji} for all j such that $1 \leq i \neq j \leq \ell$.

Finally, it can be shown that the graph $G = G(p, k, \ell)$ constructed above has a unique separable p-regular k-clique cover unless $\ell = 4$ and $k = 3p$, when there are exactly two such covers. This exceptional case is most easily illustrated when $p = 1$. The underlying graph is $K_{2,2,2}$, which is the 1-skeleton of the octahedron O_3. One 3-clique cover is formed by four of the octahedron's triangular faces, no two of which share an edge; the other cover is formed by the remaining four faces. The case for general p is just the composition $O_3[K_p]$, which is also described as K_{6p} minus 3 vertex-disjoint copies of $K_{p,p}$. We leave the details to the reader. ■

Observe that for $k = (\ell - 1)p$, the graph $G = G(p, k, \ell)$ in the above theorem is isomorphic to the line graph of a complete multigraph of order ℓ in which each pair of vertices is joined by p parallel edges.

3 Regular clique covers

We continue with the more general problem of determining (or at least estimating) the smallest order $\mu(p, k, \ell)$ of a graph that admits a p-regular k-clique cover with exactly ℓ cliques. Trivially, for $\ell = 1$, $\mu(p, k, 1) = k$. It is also easy to see that if $\ell = 2$, then $\mu(p, k, 2) = 2k - p$. From now on we therefore assume that $\ell \geq 3$.

We first examine the case when ℓ is small in terms of k/p.

Proposition 2. *Let p, k, ℓ be positive integers with $p < k$ and $\ell \geq 3$. If $k \geq (\ell - 1)p$, then $\mu(p, k, \ell) = k\ell - p\binom{\ell}{2}$.*

Proof. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_\ell\}$ be a family of k-cliques in a graph G of order n, which forms a p-regular k-clique cover of G. By the inclusion-exclusion principle we have $n \geq \sum_{1 \leq i \leq \ell} |V(H_i)| - \sum_{1 \leq i < j} |V(H_i) \cap V(H_j)| = k\ell - p\binom{\ell}{2}$. On the other hand, the graphs $G(p, k, \ell)$ constructed in Theorem 1 have order equal to the lower bound just obtained. ∎

For values of ℓ that are larger in terms of k/p we focus on the case $p = 1$. This is motivated in part by the following lemma.

Lemma 3. *Let p, k, ℓ be positive integers with $p < k$ and $\ell \geq 3$. Then*

$$\mu(p, k, \ell) \leq \mu(1, k - p + 1, \ell) + (p - 1).$$

Proof. Let G be a graph that achieves the bound for $\mu(1, k - p + 1, \ell)$. We form G' by adding $p - 1$ new vertices $\infty_1, \ldots \infty_{p-1}$ that are adjacent to all vertices of G. Each of the ℓ distinguished cliques of G of size $k - p + 1$ corresponds to a distinguished clique of G' of size k. Any two such cliques of G intersect in a single vertex, hence any two corresponding cliques of G' intersect in p vertices, that vertex and the $p - 1$ ∞_i's. It follows that $\mu(p, k, \ell) \leq \mu(1, k - p + 1, \ell) + (p - 1)$ as desired. ∎

The next lemma shows that no vertex lies in too many cliques in a 1-regular cover.

Lemma 4. *Suppose that a vertex v lies in k different k-cliques in a 1-regular cover. Then v lies in all k-cliques.*

Proof. By way of contradiction, suppose that there is a k-clique C not containing v. Then C must intersect each of the k cliques containing v. Since no two cliques share more than one vertex, these k points of intersection contain every vertex of C. It follows that v is adjacent to each vertex in C. Hence C is not a clique, which is our desired contradiction. ■

Using Lemma 4 we can determine $\mu(1, k, \ell)$ exactly for fixed k and large values of ℓ.

Proposition 5. *Suppose that $k \geq 2$ and $\ell \geq (k-1)^2+1$. Then $\mu(1, k, \ell) = \ell(k-1)+1$.*

Proof. Fix a clique C. Then each of the (at least) $(k-1)^2 = k(k-2)+1$ of the remaining cliques intersect C in a vertex. It follows that one of the k points in C, say v is in at least $k-1$ remaining cliques. Including C, v lies in at least k cliques. By Lemma 4, v is in all cliques. So for any two cliques all other vertices are distinct, and the graph has at least $\ell(k-1)+1$ vertices as claimed. The graph formed by gluing ℓ cliques along a single vertex has the desired clique cover and the correct order, so equality holds. ■

By Propositions 2 and 5, respectively, we have the exact value for $\mu(1, k, \ell)$ for $\ell \leq k+1$ and $\ell \geq (k-1)^2+1$. To examine the remaining cases we show an interesting relation with design theory. Note that this relation holds for general p.

A *block design* D is a set $\mathcal{V}$ of *varieties* together with a collection $\mathcal{B}$ of subsets of $\mathcal{V}$, called *blocks*, such that every pair of varieties occurs together in exactly λ blocks. We will call D a (v, k, λ)-design if all of the blocks are of cardinality k and $v = |\mathcal{V}|$. For example, a *projective plane* is a $(n^2+n+1, n+1, 1)$-design, and an *affine plane* is a $(n^2, n, 1)$-design.

The *replication number* $r(v)$ of a variety v is the number of blocks incident with v. Say R is a *replication bound* if $r(v) \leq R$ for all varieties v. If all blocks are of the same cardinality, then $r(v)$ is constant over all v. A simple counting argument shows that $R = \lambda(v-1)/(k-1)$ is a replication bound.

A block design D (with possibly varying block sizes) is a **-design* if no block that does not contain a variety v intersects all blocks that do contain v.

The following theorem relates regular clique covers and *-designs. To avoid confusion we add primes to all parameters associated with the *-design.

Theorem 6. *There exists a graph G on n vertices with a p-regular k-clique cover with ℓ cliques if and only if there exists a *-design D' with replication bound $R' = k$ and $v' = \ell$, $\lambda' = p$, and $n = |\mathcal{B}| + \sum_{u' \in \mathcal{V}'}(R' - r(u'))$.*

Proof. The proof uses a type of duality. Suppose that we are given a graph G with the desired clique covering. Then form the design D' by setting the varieties equal to the set of cliques in the clique-cover, and for each vertex $u \in V(G)$ forming a block B' on the varieties corresponding to the cliques containing u. In G, the number of cliques containing a given vertex can vary, so in D' the cardinality of the blocks can vary. However, in G all of the cliques are of the same size, hence in D' the replication number is a constant $R' = k$. Given any two varieties, their corresponding cliques intersect in exactly p vertices, hence these varieties lie together in exactly $\lambda' = p$ blocks. Similarly, since there is no vertex u incident to all vertices in a clique of G, there is no block that intersects all blocks on a given variety. More simply, the fact that these complete subgraphs in G are cliques corresponds to the *-property on the design.

Conversely, suppose that we are given a design D' with replication bound R'. We can add singleton blocks to D' until each variety appears in exactly R' blocks. Note that this does not change the property that every pair of varieties appears together in exactly λ' blocks, nor does it change the value $n = |\mathcal{B}| + \sum_{u' \in \mathcal{V}'}(R' - r(u'))$, or the property of being a *-design. We now reverse the above construction: form G whose vertices are the blocks of the design D' (after adding the singleton blocks), and joining two vertices with an edge whenever the corresponding blocks have non-empty intersection. The constant replication number R' gives a collection of complete subgraphs that together contain all edges. The *-condition implies that these subgraphs are cliques. The number of vertices in common in two cliques is the same as the number of blocks containing the corresponding pair of varieties, i.e., $p = \lambda'$ as claimed. ■

The preceding theorem can be awkward to use in practice, so we offer the following corollary.

Corollary 7. *Suppose that there exists a* (v', k', λ') **-design* D'*. Then*

$$\mu(\lambda', \lambda'(v'-1)/(k'-1), v') \leq \frac{\lambda' v'(v'-1)}{k'(k'-1)}.$$

Proof. An easy counting argument shows that D' has a replication bound $R' = \lambda'(v'-1)/(k-1)$. The result follows from Theorem 6. ■

A natural class of designs with this property are the affine planes: $(n^2, n, 1)$-designs. These are easily seen to be *-designs, since the set of blocks on a variety includes one block from each parallel class, and any other block lies in some parallel class and hence misses one block on that variety. Affine planes are known to exist for n a prime power. These give the following.

Corollary 8. *When* q *is a prime power*

$$\mu(1, q+1, q^2) = q(q+1).$$

Proof. We use the affine plane with $n = q$ and apply Corollary 7 to get the upper bound on $\mu(1, q+1, q^2)$. For the lower bound there are two cases. Fix a clique C. If one vertex of C is in at least $q+1$ cliques, then by Lemma Proposition 5 it lies in all cliques. A counting argument similar to that of Proposition 5 shows that the number of vertices is at least $q^3 + 1$. If each vertex of C lies in at most q cliques, then counting vertex-clique incidences in two ways shows that the number of vertices is at least $q(q+1)$. ■

The preceding corollary shows that the bound on ℓ in Proposition 5 is the best possible.

4 Concluding remarks

In this paper we have laid the foundations for studying $\mu(p, k, \ell)$, but there are still many interesting special cases. We examine some of these here.

For arbitrary p and $\ell \leq k/p+1$ we have the exact results as given in Lemma 2. For larger ℓ, we have some special results in the case $p=1$. It would be nice to generalize Lemma 4 and Proposition 5 to values $p>1$.

For $p=1$ we have the exact values of μ for ℓ outside the range $k+2 \leq \ell \leq (k-1)^2$. However, we do not have good estimates within this range. It is clear that for $\ell \leq (k-1)^2$ we have $\mu(1,k,\ell) \leq 1+\ell(k-1) \leq 1+(k-1)^3 = \Theta(k^3)$. We conjecture that these values are $O(k^2)$.

Conjecture 9. *There exists a constant c such that for any k and any $\ell \leq (k-1)^2$, we have $\mu(1,k,\ell) \leq ck^2$.*

However, we have not been able to make much progress on this conjecture. Using Theorem 6 we would need classes of designs with a bounded replication number on the order of square root of the number of varieties. Most research in design theory focus on constant block sizes. For these designs the replication number grows linearly in the number of varieties. This leads to an improvement on the coefficient of the k^3 term, but does not address Conjecture 9.

The examples based on affine planes suggest that for ℓ close to $(k-1)^2$, we need block designs that are in a sense close to affine planes. This is generally considered a difficult problem.

The parameter $\mu(p,k,\ell)$ is trivially monotone in ℓ. That is, if $\ell \leq \ell'$, then $\mu(p,k,\ell) \leq \mu(p,k,\ell')$. We do not know if similar monotone properties hold for k and for p. One would expect larger clique sizes k to correspond to a larger number of vertices. This intuition is incorrect past a certain point: if k is too large compared to ℓ, then Proposition 5 applies. Avoiding this case, we conjecture the following.

Conjecture 10. *If $k \leq k'$ and $\ell \leq (k-1)^2$, then $\mu(p,k,\ell) \leq \mu(p,k',\ell)$.*

Similarly, one would expect that if the clique intersection was larger, then we would have fewer vertices. Hence we conjecture:

Conjecture 11. *If $p \leq p'$, then $\mu(p,k,\ell) \geq \mu(p',k,\ell)$.*

Finally, our results are related to a well-known result of Erdös, Ko and Rado (see, e.g., [26]) : If $n \geq (p+1)(k-p+1)$ and $\mathcal{F}$ is a family of k-subsets of an n-set so that any two members of $\mathcal{F}$ meet in at least p points, then $|\mathcal{F}| \leq \binom{n-p}{k-p}$.

Based on this result, given the parameters $\ell \geq 3$ and $k > p$, the size n of a minimal graph G p-regularly covered by a family of ℓ k-cliques must be at least as big as to satisfy the inequality

$$\ell \leq \binom{n-p}{k-p}.$$

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