On stability of the hamiltonian index under contractions and closures

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Abstract

The hamiltonian index of a graph G is the smallest integer k such that the k-th iterated line graph of G is hamiltonian. We first show that, with one exceptional case, adding an edge to a graph cannot increase its hamiltonian index. We use this result to prove that neither the contraction of an $A_G(F)$-contractible subgraph F of a graph G nor the closure operation performed on G (if G is claw-free) affects the value of the hamiltonian index of a graph G.

Keywords: hamiltonian index, stable property, closure of a graph, contractible graph, collapsible graph

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1 Introduction

In this paper, we consider only finite undirected loopless graphs $G = (V(G), E(G))$. However, except for Section 4, we admit G to have multiple edges. We generally follow the most common graph-theoretical notation and terminology and for concepts and notations not defined here we refer the reader to [1].

A *dominating closed trail* (abbreviated DCT) in a graph G is a closed trail (or, equivalently, an eulerian subgraph) T in G such that every edge of G has at least one vertex on T. The following result by Harary and Nash-Williams relates the existence of a DCT

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in a graph G and the existence of a hamiltonian cycle in its line graph $L(G)$. Here the line graph of a graph G, denoted by $L(G)$, is the graph with vertex set $E(G)$ and with two vertices adjacent in $L(G)$ if and only if the corresponding edges of G have a vertex in common.

Theorem A [5]. *Let G be a graph with at least three edges. Then $L(G)$ is hamiltonian if and only if G has a DCT.*

If $P = x_1, \ldots, x_k$ is a path in a graph G and $S, T \subset G$ are subgraphs of G, then we say that P is an *(S,T)-path* if $x_1 \in V(S)$ and $x_k \in V(T)$. The *distance* of two subgraphs $S, T \subset G$ (denoted $\mathrm{dist}_G(S,T)$) is the minimum length of an (S,T)-path. For any integer $i \geq 0$ set $V_i(G) = \{v \in V(G) : d_G(v) = i\}$ (here $d_G(v)$ denotes the degree of a vertex v in G) and $W(G) = V(G) \setminus V_2(G)$. *A branch* in G is a nontrivial path with endvertices in $W(G)$ and with internal vertices, if any, of degree 2 in G (and thus not in $W(G)$). If a branch has length 1, then it has no internal vertex. Let $B(G)$ denote the set of branches of G, and let $B_1(G)$ be the subset of $B(G)$ in which every branch has an end in $V_1(G)$. For any subgraph H of G let $B_H(G)$ be the set of those branches of G which have all edges in H.

If G is a graph and $k \geq 2$ an integer, then $EU_k(G)$ denotes the set of all subgraphs H of G that satisfy the following conditions:

(I) $d_H(x) \equiv 0 \pmod 2$ for every $x \in V(H)$;

(II) $V_0(H) \subseteq \bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H)$;

(III) $\mathrm{dist}_G(H_1, H - H_1) \leq k - 1$ for every subgraph H_1 of H;

(IV) $|E(b)| \leq k + 1$ for every branch $b \in B(G) \setminus B_H(G)$;

(V) $|E(b)| \leq k$ for every branch $b \in B_1(G)$.

The following theorem, which can be considered as an analogue of Theorem A for the k-th iterated line graph $L^k(G)$ of a graph G, shows the importance of subgraphs from $EU_k(G)$. Here $L^k(G)$ is defined recursively by $L^0(G) = G, L^1(G) = L(G)$ and $L^k(G) = L(L^{k-1}(G))$.

Theorem B [13]. *Let G be a connected graph with at least three edges and let $k \geq 2$ be an integer. Then $L^k(G)$ is hamiltonian if and only if $EU_k(G) \neq \emptyset$.*

The *hamiltonian index* of a graph G, denoted by $h(G)$, is the smallest integer k such that the k-th iterated line graph $L^k(G)$ of G is hamiltonian. Thus, Theorem B equivalently says that for an integer $k \geq 2$ and for any graph G, $h(G) \leq k$ if and only if $EU_k(G) \neq \emptyset$.

If F is a subgraph of a graph G, then a vertex x is said to be a *vertex of attachment of F in G* if $x \in V(F)$ and x has a neighbor in $V(G) \setminus V(F)$. The set of all vertices of attachment of a subgraph F in G is denoted by $A_G(F)$.

For a subgraph F of G, $G|_F$ denotes the graph obtained from G by identifying the vertices of F as a (new) vertex v_F, and by replacing the created loops by pendant edges (i.e. edges with one vertex of degree 1) attached to v_F. We say that the graph $G|_F$ is obtained from G by *contracting* the subgraph F (observe that $|E(G)| = |E(G|_F)|$).

If G is a graph, $X \subset V(G)$ and $\mathcal{A}$ is a partition of X into subsets, then $E(\mathcal{A})$ denotes the set of all edges a_1a_2 (not necessarily in $E(G)$) such that a_1, a_2 are in the same element of $\mathcal{A}$, and $G^{\mathcal{A}}$ denotes the graph with vertex set $V(G^{\mathcal{A}}) = V(G)$ and edge set $E(G^{\mathcal{A}}) = E(G) \cup E(\mathcal{A})$. Note that $E(G)$ and $E(\mathcal{A})$ are considered to be disjoint, i.e., if $e_1 = a_1a_2 \in E(G)$ and $e_2 = a_1a_2 \in E(\mathcal{A})$, then e_1, e_2 are parallel edges in $G^{\mathcal{A}}$.

Let F be a graph and let $A \subset V(F)$. Following [11], we say that the graph F is *A-contractible*, if for every even subset $X \subset A$ and for every partition $\mathcal{A}$ of X into two-element subsets the graph $F^{\mathcal{A}}$ has a DCT containing all vertices of A and all edges of $E(\mathcal{A})$. Note that this definition comprises the case where X is empty and $F^{\mathcal{A}} = F$. Also, if F is A-contractible, then F is A'-contractible for any $A' \subset A$ (since every subset X of A' is a subset of A).

Set $d_T(G) = \max\{\ |S| : S \subset E(G)$ and there is a closed trail $T \subset G$ such that every edge $e \in S$ has at least one vertex on $T\}$. The following result was proved in [11].

Theorem C [11]. *Let F be a connected graph and let $A \subset V(F)$. Then F is A-contractible if and only if*

$$d_T(G) = d_T(G|_F)$$

for every graph G such that $F \subset G$ and $A_G(F) = A$.

For $d_T(G) = |E(G)|$ we get the following immediate corollary.

Corollary D [11]. *Let G be a graph and let $F \subset G$ be an $A_G(F)$-contractible subgraph of G. Then G has a DCT if and only if $G|_F$ has a DCT.*

Note that $G|_F$ may contain multiple edges even if G is a simple graph. However, it is easy to observe that a multiple edge is a contractible subgraph and hence, by a series of subsequent contractions, it is always possible to reduce $G|_F$ to a certain simple graph G' with $d_T(G') = d_T(G|_F) = d_T(G)$.

We say that a graph G is *claw-free* if G is a simple graph that does not contain a copy of the *claw* as an induced subgraph. It is well-known that every line graph is claw-free.

Let G be a claw-free graph. A vertex $x \in V(G)$ is *locally connected* if $G[N(x)]$ is a connected graph. For $x \in V(G)$, the graph G'_x with vertex set $V(G'_x) = V(G)$ and edge set $E(G'_x) = E(G) \cup \{yz|\ y, z \in N(x)\}$ is called the *local completion of G at x*. It was shown in [9] that the local completion of a claw-free graph G at x is again claw-free, and if x is a locally connected vertex, then $c(G'_x) = c(G)$, where $c(G)$ denotes the circumference of G, i.e. the length of a longest cycle in G.

The following concept was introduced in [9]. Let G be a claw-free graph and let $\mathrm{cl}(G)$ be a graph obtained from G by recursively performing the local completion operation at locally connected vertices with noncomplete neighborhood, as long as this is possible. The graph $\mathrm{cl}(G)$ is called the *closure* of the graph G. The following theorem summarizes basic properties of the closure operation.

Theorem E [9]. *Let G be a claw-free graph. Then*

(*i*) $\mathrm{cl}(G)$ *is uniquely determined,*

(*ii*) $c(\mathrm{cl}(G)) = c(G)$,

(*iii*) $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph.*

Theorem E has the following immediate consequence.

Corollary F [9]. *Let G be a claw-free graph. Then G is hamiltonian if and only if* $\mathrm{cl}(G)$ *is hamiltonian.*

If $\mathcal{C}$ is a class of graphs, Γ is a graph operation on $\mathcal{C}$ and $\mathcal{P}$ is a graph property, then $\mathcal{P}$ is said to be *stable under* Γ if, for any $G \in \mathcal{C}$, G has $\mathcal{P}$ if and only if $\Gamma(G)$ has $\mathcal{P}$. Similarly, a graph invariant π is said to be *stable under* Γ if for any $G \in \mathcal{C}$ we have $\pi(G) = \pi(\Gamma(G))$. In this terminology, Theorem C and Corollary D say that $d_T(G)$ and the existence of a DCT are stable under the operation of contraction of an $A_G(F)$-contractible subgraph F, and Theorem E and Corollary F say that the circumference and hamiltonicity are stable under the closure operation on claw-free graphs. Stability of some further graph properties and invariants under the closure operation was studied e.g. in [2], [10], [6] or [8] (see also the survey paper [3]).

The main results of this paper, Theorems 7 and 10, show that the hamiltonian index is stable under the operation of contraction of an $A_G(F)$-contractible subgraph F and under the closure operation on claw-free graphs.

2 The hamiltonian index of a subgraph

Our first result shows that, with one exceptional case, adding an edge to a graph cannot increase its hamiltonian index.

Theorem 1. *Let G be a connected graph with at least three edges that is not a path. Then for any two vertices $a, b \in V(G)$ with $d_G(a) + d_G(b) \geq 3$, either $h(G) = 1$ and $h(G + ab) = 2$ or $h(G) \geq h(G + ab)$. Moreover, if $\mathrm{dist}_G(a, b) = 2$, then*

$$h(G) \geq h(G + ab).$$

Proof. Let $G' = G + ab$. We distinguish the following cases.

<u>Case 1:</u> $h(G') = 0$. Then $h(G) \geq 0 = h(G')$.

<u>Case 2:</u> $h(G') = 1$. Then G' is not hamiltonian, implying that G is also not hamiltonian. Hence $h(G) \geq 1 = h(G')$.

<u>Case 3:</u> $h(G') \geq 2$.
If $h(G) = 0$, then G is hamiltonian and since $V(G) = V(G')$, we have $h(G') = 0$, a contradiction.

Next, suppose $h(G) = 1$. Then, by Theorem A, G has a DCT T. Since $h(G') \geq 2$, T is not a DCT of G'. Hence neither a nor b are in $V(T)$, and necessarily all neighbors of a and all neighbors of b are on T. This implies that any hamiltonian cycle in $L(G)$ is a DCT in $L(G')$, implying that $h(G') \leq 2$. Since, by the assumption, $h(G') \geq 2$, we have $h(G) = 1$ and $h(G') = 2$.

Now, for $a, b \in V(G)$ with $\mathrm{dist}_G(a, b) = 2$, neither a nor b are in $V(T)$ and hence there is a vertex c_{ab} in $N_G(a) \cap N_G(b)$ with $c_{ab} \in V(T)$. Let T' be a closed trail in G' obtained from T by adding the cycle $c_{ab}abc_{ab}$. Then T' is a DCT in G', implying $h(G') \leq 1$, a contradiction.

Hence we can suppose that $h(G) \geq 2$ and $d_G(a) + d_G(b) \geq 3$. By Theorem B, there is a subgraph $H \in EU_{h(G)}(G)$. Let H' be the subgraph of G' with vertex set

$$V(H') = V(H) \cup \{v \in \{a, b\} : d_{G'}(v) \geq 3\}$$

and edge set

$$E(H') = E(H).$$

We will show that $H' \in EU_{h(G)}(G')$, i.e., H' satisfies the conditions (I) – (V) of the definition of $EU_{h(G)}(G')$ for the graph G' and $k = h(G)$. Obviously, H' satisfies conditions (I) and (II).

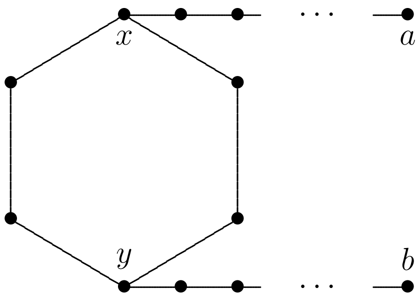
If one of a, b has degree 1 in G, say, $d_G(a) = 1$, then $d_G(b) \geq 2$ since $d_G(a) + d_G(b) \geq 3$. The branch P of $B_1(G)$, which contains a, will become a new branch $P' = Pb$ in $B(G') \setminus (B_{H'}(G') \cup B_1(G'))$ of length $|E(P)| + 1 \leq h(G) + 1$. The other branches of $B(G') \setminus B_{H'}(G)$ are the same as those of $B(G) \setminus B_H(G)$ except when $d_G(b) = 2$ and b is not in $V(H)$; in this exceptional case, the branch containing b turns into two shorter branches in $B(G') \setminus B_{H'}(G')$. This shows that H' satisfies (IV) and (V). If both a and b have degree at least 2 in G, then the branches in $B(G') \setminus B_{H'}(G')$ are the same as those in $B(G) \setminus B_H(G)$ except when a or b (or both) have degree exactly 2 in G and they are not in $V(H)$; in this exceptional case, the branches in $B(G') \setminus B_{H'}(G')$ will be shorter than those in $B(G) \setminus B_H(G)$. This shows that H' satisfies (IV) and (V).

It remains to show that H' satisfies (III). Suppose there is a subgraph H'_1 of H' such that $\mathrm{dist}_{G'}(H'_1, H' - H'_1) \geq h(G) \geq 2$. It is easy to see that $V(H'_1) \cap V(H)$ and $V(H' - H'_1) \cap V(H)$ cannot be both empty. Suppose first that $V(H'_1) \cap V(H) = \emptyset$ and $V(H' - H'_1) \cap V(H) \neq \emptyset$ (note that the case that $V(H'_1) \cap V(H) \neq \emptyset$ and $V(H' - H'_1) \cap V(H) = \emptyset$ is symmetric). Then $V(H'_1) \subseteq \{a, b\}$. Let x be a vertex of H'_1. Since $V(H'_1) \cap V(H) = \emptyset$, $d_G(x) \leq 2$ due to (II). But by the definition of H', $d_{G'}(x) \geq 3$, hence $d_G(x) = 2$ and x belongs to a branch in $B(G) \setminus B_H(G)$. Since H satisfies (IV) and (V), $\mathrm{dist}_G(\{x\}, H) \leq h(G) - 1$. Now, every shortest path from $V(H'_1)$ to H in G is also an $(H'_1, H' - H'_1)$-path in G' which implies $\mathrm{dist}_{G'}(H'_1, H' - H'_1) \leq \mathrm{dist}_G(\{x\}, H) \leq h(G) - 1$, a contradiction. This implies that H'_1 has exactly one vertex, say, $V(H'_1) = \{a\}$. Similarly, $\mathrm{dist}_G(\{a\}, H) \leq h(G) - 1$ and any shortest $(\{a\}, H)$-path in G is an $(H'_1, H' - H'_1)$-path in G', implying that $\mathrm{dist}_{G'}(H'_1, H' - H'_1) \leq \mathrm{dist}_G(\{a\}, H) \leq h(G) - 1$, a contradiction. Finally, suppose that both $V(H'_1) \cap V(H)$ and $V(H' - H'_1) \cap V(H)$ are nonempty, and set $H_1 = H'_1 \cap H$. Analogously as above, any shortest $(H_1, H - H_1)$-path in G is also an $(H'_1, H' - H'_1)$-path in G'. Hence $\mathrm{dist}_{G'}(H'_1, H' - H'_1) \leq \mathrm{dist}_G(H_1, H - H_1) \leq h(G) - 1$, a contradiction. This shows that H' satisfies (III). Thus $H' \in EU_{h(G)}(G')$, implying $h(G') \leq h(G)$.

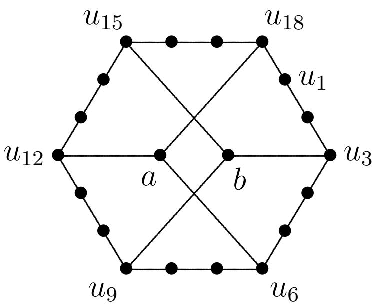
If $\mathrm{dist}_G(a, b) = 2$ and $d_G(a) + d_G(b) = 2$, then both a and b are on branches of length 1 which are all in $B_1(G)$. It is obvious that $h(G) = 1$ implies $h(G') = 1$. If $h(G) \geq 2$, then every member of $EU_{h(G)}(G)$ is also a member of $EU_{h(G)}(G')$, thus $h(G') \leq h(G)$. ■

Example 2. We construct an infinite family of graphs showing that the assumption $d_G(a) + d_G(b) \geq 3$ in Theorem 1 cannot be relaxed. Let C be a cycle of length $|E(C)| \geq 6$

and let x, y be two vertices on C with maximum $\mathrm{dist}_C(x, y)$. Take two disjoint paths P_1, P_2 with endvertices x', a and y', b, respectively. Let G be the graph obtained from C and P_1, P_2 by identifying x', x and y', y respectively (for $|E(C)| = 6$ see Figure 1(a)). It is easy to see that P_1 and P_2 are two branches in $B_1(G)$. If $|E(P_1)|, |E(P_2)| \le (|E(C)|-2)/4$, then $h(G) = \max\{|E(P_1)|, |E(P_2)|\}$ (see [12] and [13]) and $h(G + ab) = |E(P_1)| + |E(P_2)| = h(G) + \min\{|E(P_1)|, |E(P_2)|\} > h(G)$ (see [12] and [14]).



(a)



(b)

Figure 1

Remark 3. In fact, using the method of the proof of Theorem 1 (with just a slight modification of the proof of (IV) and (V)), it would be possible to show that without the assumption $d_G(a) + d_G(b) \ge 3$ one can still prove that $2h(G) \ge h(G + ab)$. The graph G from Example 2 with $|E(P_1)| = |E(P_2)| \le (|E(C)| - 2)/4$ gives $2h(G) = h(G + ab)$ (see [14]), which shows that this inequality is sharp.

Using a similar modification of the proof of Theorem 1, it would also be possible to prove that $2h(G) \ge h(G')$ if G is a spanning subgraph of G'. Details are left to the reader.

Example 4. Without the condition $\mathrm{dist}_G(a, b) = 2$, we can construct a graph G such that $h(G) = 1$ and $h(G + ab) = 2$ even if $d_G(a) + d_G(b) \ge 3$ is required. Let $t, s \ge 3$ be integers, let $C = u_1u_2 \cdots u_t \cdots u_{2t} \cdots u_{st}u_1$ be a cycle of length st and let a and b be two distinct vertices that are not on C. The graph G is obtained from C and a, b by adding s new edges between a, b and $u_t, u_{2t}, \cdots u_{st}$ such that each of a, b is incident to at least one and each of $u_t, u_{2t}, \cdots, u_{st}$ is incident to exactly one of the new edges (for $t = 3$, $s = 6$ and one of the possible choices of the new edges see Figure 1(b)). By the construction, $d_G(a) + d_G(b) = s \ge 3$. It is easy to see (by Theorem A) that $h(G) = 1$ and $h(G+ab) = 2$.

The following corollary is easily obtained from Theorem 1.

Corollary 5. *Let G be a connected graph with at least three edges that is not a path and let G' be a graph obtained from G by recursively adding the edges whose ends a and b satisfy the assumptions of the first part of Theorem 1. Then either $h(G) = 1$ and $h(G') = 2$, or $h(G) \ge h(G')$.*

3 The hamiltonian index is stable under contraction

We begin this section with the following easy observation which will be used in our proof.

Lemma 6. *Let G be a graph with $h(G) \geq 2$. For any $H \in EU_{h(G)}(G)$ and any subgraph H_1 of H, if the distance between H_1 and $H - H_1$ is at least 2, then the shortest path of G between H_1 and $H - H_1$ is a branch of G, whose ends are not adjacent in G.*

Proof. The lemma follows easily from the condition (II) of the definition of $EU_{h(G)}(G)$. ■

We will also need the following well-known result.

Theorem G [7]. *A connected graph is eulerian if and only if each minimum edge cut contains an even number of edges.*

If G is a hamiltonian graph (i.e. $h(G) = 0$) and $F \subset G$ is a nontrivial subgraph of G, then $G|_F$ cannot be hamiltonian (since it has connectivity 1), and it is easy to observe that any hamiltonian cycle in G turns into a DCT in $G|_F$. Hence $h(G) = 0$ implies $h(G|_F) = 1$ for any nontrivial subgraph $F \subset G$. However, the following theorem shows that for $h(G) \geq 1$, i.e. for nonhamiltonian graphs, the hamiltonian index is stable under contraction of a contractible subgraph.

Theorem 7. *Let G be a nonhamiltonian graph other than a path and F be an $A_G(F)$-contractible subgraph of G. Then $h(G) = h(G|_F)$.*

Proof. Let $G' = G|_F$. By Theorem A and Corollary D, $h(G) \leq 1$ if and only if $h(G') \leq 1$. Equivalently, $h(G) \geq 2$ if and only if $h(G') \geq 2$. It is sufficient to consider the case $h(G) \geq 2$. We first prove that $h(G') \leq h(G)$. By Theorem B and $h(G) \geq 2$, we can take a subgraph H in $EU_{h(G)}(G)$. Let H' be the graph obtained from $H|_F$ by deleting the new pendant edges. We shall prove that H' is in $EU_{h(G)}(G')$, i.e., that H' satisfies the conditions of the definition of $EU_{h(G)}(G')$ for the graph G' and $k = h(G)$. It is easy to see that H' satisfies the conditions (I) and (II) due to Theorem G.

The following claim is immediate from the definitions of $A_G(F)$ and A-contractible graph.

Claim 1. Every vertex in $A_G(F)$ has degree at least 3 in G.

Now Claim 1 and Lemma 6 easily imply that H' satisfies also the other conditions in the definition of $EU_{h(G)}(G')$, and hence $h(G') \leq h(G)$.

We will prove that $h(G) \leq h(G')$. Since $h(G') \geq 2$, by Theorem B, we can take a subgraph H' in $EU_{h(G')}(G')$. Obviously, every edge of H' can be considered as an edge of G. Set $V_b(H') = \{x \in F : x$ is an endvertex of a branch of $B_{H'}(G)\}$ and let $r(x)$ denote the number of branches of $B_{H'}(G)$ which have x as an endvertex. Set $V_b^j = \{x \in V_b(H') : r(x) \equiv j \pmod 2\}$. Since H' satisfies (I), $\sum_{x \in V_b^1} r(x) + \sum_{x \in V_b^2} r(x) = \sum_{x \in V_b} r(x) = d_{H'}(v_F)$ is even. But $\sum_{x \in V_b^2} r(x)$ is even, hence $\sum_{x \in V_b^1} r(x)$ is also even, which implies that $|V_b^1|$ is even. Let $X = V_b^1$ and take one partition $\mathcal{A}$ of X into two-element subsets. Since F

is $A_G(F)$-contractible, $F^{\mathcal{A}}$ has a DCT T containing all vertices of $A_G(F)$ and all edges of $E(\mathcal{A})$. Now we let H be the graph with vertex set

$$V(H) = V(H') \cup (\bigcup_{i=3}^{\Delta(G)} V_i(G)) \cup V(T)$$

and edge set

$$E(H) = E(H') \cup (E(T) \setminus E(\mathcal{A})).$$

We prove that $H \in EU_{h(G')}(G)$. Obviously, H satisfies the conditions (I) and (II) in the definition of $EU_{h(G')}(G)$. Since T is a DCT which contains all vertices of $A_G(F)$ and all edges of $E(\mathcal{A})$, by Claim 1, H satisfies (IV) and (V). By Lemma 6, H satisfies (III). Hence $H \in EU_{h(G')}(G)$, implying $h(G) \leq h(G')$. This completes the proof of Theorem 7. ■

Remark 8. Catlin [4] introduced a reduction technique based on the concept of a collapsible graph. It was shown in [11] that every collapsible graph F is $V(F)$-contractible. Thus, Theorem 7 implies that the hamiltonian index is stable under contraction of a collapsible subgraph.

4 The hamiltonian index of a claw-free graph is stable under the closure

In this section we assume all graphs to be simple (i.e., without multiple edges).

Lemma 9. *Let G be a connected claw-free graph with at least three edges which is not a path. Then*
(*i*) $h(G) = 0$ *if and only if* $h(\mathrm{cl}(G)) = 0$;
(*ii*) $h(G) = 1$ *if and only if* $h(\mathrm{cl}(G)) = 1$.

Proof. By Corollary F, it is sufficient to prove that $h(G) \leq 1$ if and only if $h(\mathrm{cl}(G)) \leq 1$. Since $V(\mathrm{cl}(G)) = V(G)$, using Theorem 1 we obtain $h(\mathrm{cl}(G)) \leq h(G)$. Hence $h(G) \leq 1$ implies $h(\mathrm{cl}(G)) \leq 1$.

Conversely, suppose that $h(\mathrm{cl}(G)) \leq 1$, i.e., by Theorem A, $\mathrm{cl}(G)$ has a DCT. We prove that G also has a DCT. It is sufficient to prove that if there is a DCT in $G' = G + xy$ for any pair of vertices x and y with $xy \notin E(G)$ such that they have a common neighbor c_{xy} in G which is a locally connected vertex of G, then there is also a DCT in G. Let P be a shortest (x, y)-path in $G[N_G(c_{xy})]$. Since G is claw-free and P is chordless, $|E(P)| \leq 3$. Since $xy \notin E(G)$, $2 \leq |E(P)| \leq 3$. Let $F = G[V(P) \cup \{c_{xy}\}]$ and $F' = G'[V(P) \cup \{c_{xy}\}]$. Then F is isomorphic to the graph F_1 or F_2 and F' is isomorphic to the graph F_3 or F_4 of Figure 2.

It is easy to verify that each of the graphs F_i is $V(F_i)$-contractible, $i = 1, 2, 3, 4$. Let e be one of the pendant edges of G' adjacent to the vertex $v_{F'}$. Since $G|_F \simeq G'|_{F'} - e$ and clearly $G'|_{F'}$ has a DCT if and only if $G'|_{F'} - e$ has a DCT, by Corollary D, G' has a DCT if and only if G has a DCT. Hence the lemma follows. ■

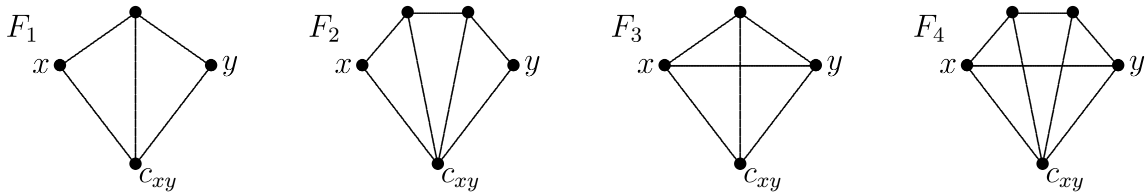


Figure 2

The following result, which is the main result of this section, shows that the hamiltonian index is stable under the closure operation in claw-free graphs.

Theorem 10. *Let G be a connected claw-free graph with at least three edges which is not a path. Then*

$$h(G) = h(\mathrm{cl}(G)).$$

Proof. By Lemma 9, we only need to prove the case when $h(G) \geq 2$. Since $G \subseteq \mathrm{cl}(G)$ and $V(G) = V(\mathrm{cl}(G))$, we have $h(G) \geq h(\mathrm{cl}(G))$ by the definition of $\mathrm{cl}(G)$ and by Theorem 1. For the reverse inequality, it is sufficient to prove that $h(G) \leq h(G + xy)$ for any pair of vertices x and y with $xy \notin E(G)$ such that they have a common neighbor in G which is a locally connected vertex of G.

Let $G' = G + xy$ and let u be a locally connected common neighbor of x and y. Then there is an (x, y)-path P in $G[N(u)]$ such that $|E(P)| \geq 2$. The following claim is immediate.

Claim 1. The internal vertices of P have degree at least 3 in G.

By Lemma 9 and since $h(G) \geq 2$, we have $h(\mathrm{cl}(G)) \geq 2$. Thus, by the definition of $\mathrm{cl}(G)$ and by Theorem 1, $h(G') \geq h(\mathrm{cl}(G)) \geq 2$. By Theorem B, $EU_{h(G')}(G') \neq \emptyset$. Taking an $H \in EU_{h(G')}(G')$, we construct a subgraph H' of G as follows:

$$V(H') = V(H) \setminus \{v \in \{x, y\} : d_G(v) = 2 \text{ and } d_H(v) = 0\},$$

$$E(H') = \begin{cases} E(H) & \text{if } xy \notin E(H), \\ (E(H)\Delta E(P)) \setminus \{xy\} & \text{if } xy \in E(H), \end{cases}$$

where $E(H)\Delta(E(P)$ denotes the symmetric difference $(E(H) \setminus E(P)) \cup (E(P) \setminus E(H))$.

We show that $H' \in EU_{h(G')}(G)$, i.e., H' satisfies the conditions of the definition of $EU_{h(G')}(G)$ for the graph G and $k = h(G')$. Obviously, H' satisfies conditions (I) and (II). By the definition of $G+xy$ and Claim 1, all branches of length at least 2 in G are the same as in G' except the case when x or y (or both) have degree 2 in G; in this exceptional case, each of x, y is on a branch in $B(G) \setminus B_1(G)$ with adjacent endvertices and length exactly 2. Hence by Claim 1 and Lemma 6, H' satisfies the other conditions of the definition of $EU_{h(G')}(G)$, implying $H' \in EU_{h(G')}(G)$. By Theorem B, $h(G) \leq h(G')$, which proves Theorem 10. ■

Remark 11. It was shown in [11] that the operation of contraction of an $A_H(F)$-contractible subgraph of a graph H can be equivalently reformulated as a closure operation performed on its line graph $G = L(H)$. Combined with the closure concept for claw-free graphs this yields a powerful closure operation on claw-free graphs, called the $\mathcal{C}$-*closure* (for details we refer the reader to [11]). Theorems 7 and 10 then immediately imply that the hamiltonian index of a claw-free graph is also stable under the $\mathcal{C}$-closure operation.

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