

# Hamilton cycles in prisms over graphs

Tomáš Kaiser<sup>1,5</sup>  
Daniel Král<sup>2,5</sup>  
Moshe Rosenfeld<sup>3</sup>  
Zdeněk Ryjáček<sup>1,5</sup>  
Heinz-Jürgen Voss<sup>4</sup>

September 30, 2005

## Abstract

The prism over a graph  $G$  is the Cartesian product  $G \square K_2$  of  $G$  with the complete graph  $K_2$ . If  $G$  is hamiltonian, then  $G \square K_2$  is also hamiltonian but the converse does not hold in general. Having a hamiltonian prism is shown to be a good measure how close a graph is to being hamiltonian. In this paper, we examine classical problems on hamiltonicity of graphs in the context of hamiltonian prisms.

## 1 Introduction

The hunt for Hamilton cycles in graphs is one of the oldest and also one of the most investigated topics in graph theory. Its origins can be traced to the search for a knight's tour on a chess board in the 9-th century through Euler's 1759 classical paper, *Solution d'une question curieuse qui ne paroit soumise a aucune analyse* (Solution of a curious question that does not seem to have been subject to any analysis), and the formal introduction of the concept by Hamilton in

---

<sup>1</sup>Department of Mathematics, University of West Bohemia, Univerzitní 8, 306 14 Plzeň, Czech Republic. E-mail: {kaisert,ryjacek}@kma.zcu.cz.

<sup>2</sup>Department of Applied Mathematics, Charles University, Malostranské nám. 25, 118 00 Praha 1, Czech Republic. E-mail: kral@kam.mff.cuni.cz.

<sup>3</sup>Computing and Software Systems Program, University of Washington, Tacoma, Washington 98402, USA. E-mail: moishe@u.washington.edu. Partial support from NSF grant INT-9802416 is gratefully acknowledged.

<sup>4</sup>Institute of Algebra, Technical University Dresden, Mommsenstrasse 13, D-01062 Dresden, Germany. E-mail: voss@math.tu-dresden.de. The author passed away in September 2003.

<sup>5</sup>Institute for Theoretical Computer Science (ITI), Charles University, Praha, Czech Republic. Supported as projects LN00A056 and 1M0021620808 of the Czech Ministry of Education.

1857. Today, there are numerous theorems, conjectures (both open and refuted), surveys and web sites dedicated to this hunt. A recent survey by Gould [14] contains a list of 248 papers on the subject that were published during the last 10 years of the 20th century.

Recent trends suggest developing measures for testing how “close” a given graph  $G$  is to being hamiltonian. One trend for instance, is to look for long cycles. As an example, consider Havel’s conjecture that the middle level of the  $(2d - 1)$ -cube is hamiltonian. Recently, Johnson [18] proved that it contains a cycle of length  $(1 - o(1))n$ . Other researchers look for related structures. A Hamilton cycle is a spanning connected 2-regular subgraph. Why not look for a spanning 3-connected cubic graph? A Hamilton cycle is a spanning closed walk in which every vertex is visited once. It is natural to relax this approach and ask for a spanning walk in which vertices may be visited more than once. A  $k$ -walk is a spanning closed walk visiting no vertex more than  $k$  times. A Hamilton cycle is then a 1-walk. Another closely related notion is that of a  $k$ -tree:  $k$ -tree is a spanning tree with all vertices of degree at most  $k$  (in particular, a 2-tree is precisely a hamiltonian path). It is not hard to show [17] that any graph with a  $k$ -tree has a  $k$ -walk, and that the existence of a  $k$ -walk guarantees the existence of a  $(k + 1)$ -tree, for any  $k$ . Hence, we have the following chain of implications:

$$1\text{-walk (Hamilton cycle)} \Rightarrow 2\text{-tree (Hamilton path)} \Rightarrow 2\text{-walk} \Rightarrow 3\text{-tree} \Rightarrow \dots$$

This suggests a “natural” hierarchy for measuring how “close” a graph is to being hamiltonian. This approach is highlighted in survey [9] by Ellingham.

The central theme of the present paper is another relaxation of hamiltonicity: the property of having a hamiltonian prism. The *prism* over a graph  $G$  is obtained by taking two copies of  $G$  and adding a perfect matching joining the two copies of each vertex by an edge. The property of having a hamiltonian prism is ‘sandwiched’ between the existence of a 2-tree and the existence of a 2-walk:

$$2\text{-tree} \Rightarrow \mathbf{\text{Hamiltonian prism}} \Rightarrow 2\text{-walk} \tag{1}$$

None of the above implications is an equivalence. In this respect, proving that a graph previously known to have a 2-walk is prism-hamiltonian is a stronger result.

Let us consider the first of the implications. If  $G$  has a Hamilton path (2-tree), then its prism is hamiltonian: just take the Hamilton path in each copy and add the two edges necessary to make a Hamilton cycle in the prism. The converse implication does not hold since the complete bipartite graph  $K_{2,4}$  has no Hamilton path and its prism is hamiltonian (see Figure 1).

As for the second implication in (1), any graph  $G$  with a hamiltonian prism has a 2-walk that follows the edges of  $G$  corresponding to the edges of the Hamilton cycle in the prism. Again, the converse does not hold as shown by the graph in

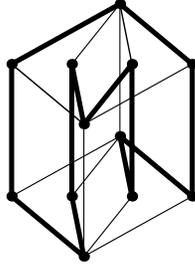


Figure 1: A Hamilton cycle in the prism over  $K_{2,4}$ .

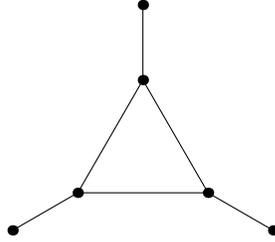


Figure 2: A graph with a 2-walk but with a non-hamiltonian prism.

Figure 2 which has a 2-walk but its prism is clearly non-hamiltonian. Similar examples can be constructed with arbitrarily large connectivity, see Section 6.

Summing up: the property of having a hamiltonian prism is properly sandwiched between being hamiltonian and the existence of a 2-walk. The initial interest in prism hamiltonicity may be traced to the attempt to tackle the conjecture of Barnette [15] that the graphs of simple 4-polytopes are hamiltonian (the conjecture is still open). Another early example of this interest are the prisms over 3-connected planar graphs. Rosenfeld and Barnette [22] showed, in 1973, that cubic planar 3-connected graphs have hamiltonian prisms if the Four Color Conjecture (open at that time) was true. Fleischner [12] found a proof avoiding the use of the Four Color Theorem. Eventually, Paulraja [21] showed that planarity is inessential here.

**Theorem 1.1.** *Any 3-connected cubic graph has a hamiltonian prism.*

Many classical questions that have been asked about the existence of Hamilton cycles or paths provide us with the opportunity to revisit and reconsider them under the prism paradigm. Indeed, each of Sections 3 to 7 of this paper is inspired by one of such classical problems. Some of our questions have been already solved in the positive, some in the negative and others still remain open. Our main interests in this paper is focused on the following classes of graphs: planar graphs (Section 3), line graphs (Section 4), 4-regular graphs (Section 5), tough graphs (Section 6), and squares of graphs (Section 7)

The spirit of this paper is encapsulated in the following example. A classical theorem of Tutte [26] states that all 4-connected planar graphs are hamiltonian. There are well-known examples of non-hamiltonian 3-connected planar graphs. Barnette [2] proved in 1967 that planar 3-connected graphs (skeletons of 3-polytopes) have a spanning 3-tree. Gao and Richter [13] showed that 3-connected planar graphs have 2-walks, that is closer to being hamiltonian in the suggested hierarchy. Can we further strengthen this result by showing that the prisms over 3-connected planar graphs are hamiltonian? Clearly, this would be the best possible result as the next level in our hierarchy are graphs with 2-walks and there are examples of 3-connected planar graphs without 2-walks.

**Conjecture 1.1.** *Any 3-connected planar graph is prism-hamiltonian.*

## 2 Notation and definitions

We refer to graphs that can have parallel edges to as *multigraphs*; if not said otherwise, a graph means a simple graph with no loops. The definition of the prism can also be rephrased as follows: the prism over  $G$  is the Cartesian product  $G \square K_2$  of  $G$  with  $K_2$ . We identify  $G$  with one of its two copies in  $G \square K_2$  and the two “clones” of a vertex  $v \in V(G)$  are denoted by  $v$  and  $v^*$ . The same notation is used for edges. Edges of the form  $vv^*$  are referred to as *vertical*.

There is a convenient way of representing 2-factors in  $G \square K_2$  by certain edge colorings of the graph  $G$  (a similar coloring scheme was defined in [7] in relation to hamiltonian decompositions). Any 2-factor  $F$  in  $G \square K_2$  induces a coloring of a subset of  $E(G)$  in three colors (blue, yellow and green), defined as follows. For any edge  $e \in E(G)$  (see Figure 3),

$$e \text{ is colored } \begin{cases} \text{blue (drawn as a dotted line)} & \text{if } F \text{ contains } e \text{ but not } e^*, \\ \text{yellow (drawn as a dashed line)} & \text{if } F \text{ contains } e^* \text{ but not } e, \\ \text{green (a dashed-and-dotted line)} & \text{if } F \text{ contains both } e \text{ and } e^*. \end{cases}$$

The subgraph  $G_{pr}$  of  $G$ , consisting of the blue, yellow and green edges derived from a 2-factor in  $G \square K_2$ , is a spanning subgraph of  $G$ . The maximum degree of a vertex in  $G_{pr}$  is 4, in which case the vertex must have 2 yellow and 2 blue edges incident with it. A vertex of degree 1 must have a single green edge incident with it, a vertex of degree 2 has either 2 green or a yellow and blue edge incident with it and a vertex of degree 3 must have a yellow, blue and green edges incident with it. If a blue-yellow-green-edge-colored graph derived from a 2-factor (or even a Hamilton cycle) is given, it is easy to reconstruct the corresponding 2-factor as illustrated in Figure 3. See also [7].

While we do not have a full characterization of all possible blue-yellow-green edge colored graphs derived from 2-factors or even Hamilton cycles in  $G \square K_2$ , we make use of a useful sufficient condition for prism-hamiltonicity from [7].

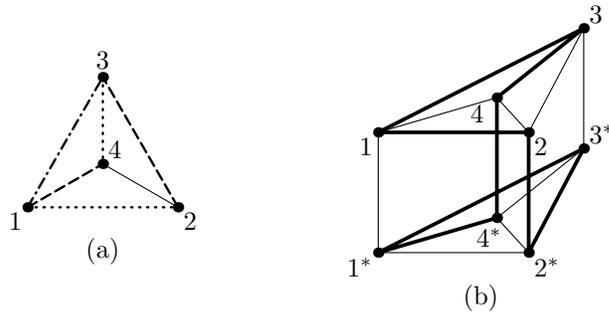


Figure 3: (a) A coloring of the complete graph  $K_4$  (an uncolored edge is shown black). (b) The corresponding Hamilton cycle (bold).

A *spanning cactus* in a graph  $G$  is a spanning connected subgraph  $H$  consisting with no two distinct cycles intersecting at a vertex. The cactus is said to be *even* if all of its cycles are of even length, i.e.,  $G$  is a bipartite graph (see Figure 4).

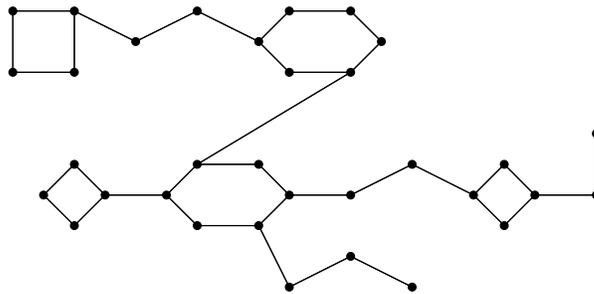


Figure 4: An even cactus.

Assume now that  $G$  has a spanning even cactus  $H$ , color the edges of the cycles blue and yellow (in an alternating way), and color all the other edges of  $H$  green. We refer the reader to [7] for an easy proof of the following proposition and for an application of it, providing an alternative proof of Theorem 1.1 (see [7]).

**Proposition 2.1** ([7]). *If  $G$  contains a spanning even cactus, then its prism is hamiltonian.*

### 3 Planar graphs

In 1946, Tutte constructed an example of a non-hamiltonian *cubic* 3-connected graph. There are infinitely many such graphs. On the other hand, *cubic* 3-connected graphs (planar or not) are prism-hamiltonian by Theorem 1.1. In fact,

cubic 3-connected planar graphs are conjectured in [1] to have a much stronger property, namely that their prism can be decomposed into two disjoint Hamilton cycles. In the case of cubic 3-connected planar *bipartite* graphs, the existence of such a decomposition was proved in [7].

We remark that Conjecture 1.1 cannot be extended to 2-connected planar graphs: for instance, the complete bipartite graph  $K_{2,n}$ ,  $n \geq 5$ , is 2-connected and planar, but its prism is not hamiltonian. Conjecture 1.1 is open even for planar triangulations. In the following subsections, we prove the conjecture for two classes of 3-connected planar graphs, namely for chordal 3-connected planar graphs (kleetopes) and for Halin graphs. Let us remark that Biebighauser and Ellingham [3] have recently extended our result on kleetops and they have shown that 3-connected triangulations of the plane, the projective plane, the torus and the Klein bottle are prism-hamiltonian.

### 3.1 Kleetopes

A *kleetope* is a plane graph obtained from a drawing of the complete graph  $K_4$  by successive subdivisions of internal faces. (A face  $F$  is *internal* if it differs from the infinite face; the subdivision of  $F$  consists in adding a new vertex  $v_F$  inside  $F$ , and joining it to all the three vertices of  $F$ .) See Figure 5a for an example of a kleetope.

Recall that a graph is *chordal* if it contains no chordless cycle of length  $\geq 4$ . It is known that kleetopes coincide with (drawings of) 3-connected chordal planar graphs. (This observation is implicit in [25, Section 2].) As a partial result in the direction of Conjecture 1.1, we shall show that 3-connected chordal planar graphs (i.e., kleetopes) are prism-hamiltonian.

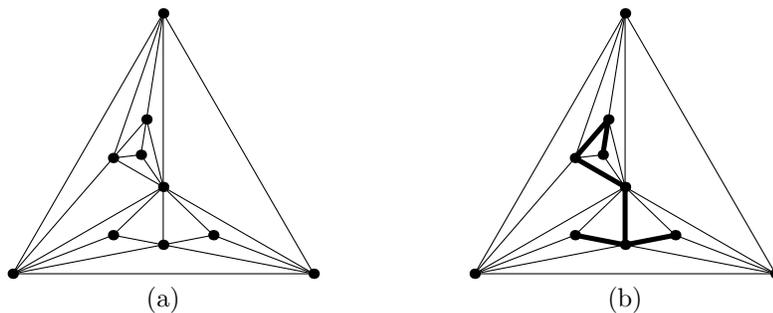


Figure 5: (a) A kleetope  $G$ . (b) The structure tree of  $G$ .

An *internal vertex* of a kleetope  $G$  is any vertex not incident with the infinite face of  $G$ . The *depth*  $\lambda(v)$  of a vertex  $v$  is defined as follows. In any drawing of  $K_4$ , the depth of the internal vertex is 0 and the depths of the other vertices are  $-1$ . If  $G$  arises from  $G'$  by subdividing an internal face  $F = x_1x_2x_3$ , then the

depth of the new vertex  $v_F$  is  $\lambda(v_F) = \max \lambda(x_i) + 1$  and all the other depths are as in  $G'$ .

Observe that in any  $G$ , there is no edge joining internal vertices of the same depth. It follows that every vertex  $v$  with  $\lambda(v) > 0$  has a *unique* neighbor  $p(v)$  (the *parent* of  $v$ ) such that  $\lambda(p(v)) = \lambda(v) - 1$ .

We may define the *structure tree*  $T(G)$  of  $G$  as the tree on all internal vertices of  $G$  such that every edge of  $T$  joins an internal vertex  $v$  (of positive depth) to its parent. (See Figure 5b.) In particular, note that if  $G$  is any drawing of  $K_4$ , then  $T(K_4)$  consists of a single vertex. In all other cases, the leaves of  $T(G)$  are precisely all internal vertices of degree 3, plus possibly the unique vertex of zero depth.

**Lemma 3.1.** *Every kleetope  $G$  can be obtained from  $K_4$  by a sequence of steps, each of which is the simultaneous subdivision of one, two or three faces containing a common internal vertex of degree 3.*

*Proof.* Consider the structure tree  $T(G)$  of a given kleetope  $G$ . We may assume  $G \neq K_4$ , so let  $m$  be a leaf of  $T(G)$  of the largest depth, and let  $u$  be the parent of  $m$ . Remove all *children* of  $u$  (the leaves whose parent is  $u$ ) to obtain a graph  $G'$  which, by the induction hypothesis, can be constructed as stated in the lemma. The vertex  $u$ , being a leaf of  $T(G')$ , is of degree 3 in  $G'$ . Subdividing all the faces of  $G'$  incident with  $u$  which correspond to the removed leaves of  $T(G)$ , we obtain the desired construction of  $G$ .  $\square$

Consider a coloring of the graph  $G$  associated to a Hamilton cycle  $C$  of the prism over  $G$ . Recall that an edge  $e$  is green in this coloring if and only if both  $e$  and  $e^*$  are in  $C$ . We shall say that the green edge  $e$  is *balanced* (in the coloring) if  $C$  traverses  $e$  and  $e^*$  in different directions.

**Theorem 3.2.** *The prism over any kleetope is hamiltonian.*

*Proof.* We prove (by induction) the stronger statement that the prism over any kleetope  $G$  contains a Hamilton cycle  $H$  such that in the associated coloring of  $G$ ,

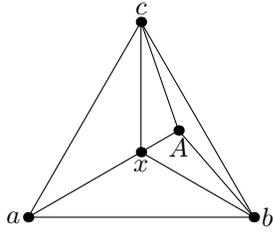
(\*) every degree 3 internal vertex  $v$  is incident with at most two colored edges, and if these are two green edges, then they are balanced.

For  $G = K_4$ , such a Hamilton cycle is shown in Figure 3b. Thus let  $G$  arise from  $G'$  by subdividing some faces sharing an internal vertex  $x$  of degree 3, as in Lemma 3.1. The neighbors of  $x$  in  $G'$  are denoted by  $a, b, c$ . We let the new vertices of  $G$  be denoted by (some of) the letters  $A, B, C$ , where  $A$  subdivides the face *not* containing  $a$ , and analogously for  $B$  and  $C$ . By symmetry, we may distinguish only 3 cases: the set  $N$  of the new vertices is  $\{A\}$ ,  $\{A, B\}$  or  $\{A, B, C\}$ . Each of the cases splits up into several subcases depending on which edges are

used by the Hamilton cycle  $H'$  of  $G'$ . We give tables indicating how the coloring is to be extended to  $G$  in each subcase (symmetric subcases omitted). The first column of the tables lists all the possible combinations of colors of the edges adjacent to  $x$  (subject to  $(*)$  and up to symmetry). To perform the modification, first uncolor all edges listed in the first column, and then apply the coloring in the second column. A path with all edges green is referred to as a green path. Similarly, a blue-yellow path has edges colored alternately blue and yellow, starting with blue.

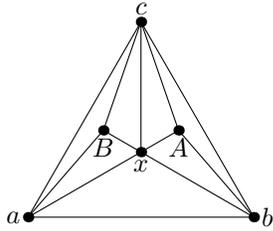
It is straightforward to check that the extended colorings do correspond to Hamilton cycles satisfying  $(*)$ .

*Case 1.* There is one new vertex  $A$ .



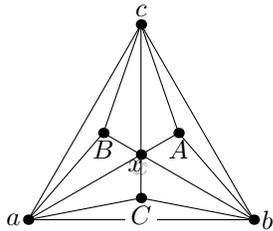
replace:	with:
green $ax$	green $axA$
green $cx$	green $cxA$
blue-yellow $axb$	blue-yellow $axb$ + green $xA$
blue-yellow $xcb$	blue-yellow $xcb$ + green $xA$
green $axb$	green $axAb$
green $xcb$	green $cxAb$

*Case 2.* There are two new vertices:  $A$  and  $B$ .



replace:	with:
green $ax$	green $aBxA$
green $cx$	green $cBxA$
blue-yellow $axb$	blue-yellow $aBxAAb$
blue-yellow $xcb$	blue-yellow $cBxAAb$
green $axb$	green $aBxAAb$
green $xcb$	green $cBxAAb$

*Case 3.* The new vertices are  $A, B$  and  $C$ . In this case, there is more symmetry and only 3 subcases to consider.



replace:	with:
green $ax$	blue-yellow $aBxCa$ + green $xA$
blue-yellow $axb$	blue-yellow $aBxAAb$ + green $xC$
green $axb$	blue-yellow $aBxAAbCa$

The last subcase of Case 3 deserves a comment. This is where the provision on balanced green edges is used. Indeed, if the edges in the green path  $axb$  were not balanced, we would obtain a disconnected 2-factor after the modification.  $\square$

### 3.2 A generalization of Halin graphs

A *Halin graph* is a plane graph such that removing all edges of its outer face  $F$ , we obtain a tree  $T$  whose leaves are precisely the vertices on  $F$ , and  $T$  has no vertices of degree 2.

We consider Halin graphs in connection to Conjecture 1.1 as an interesting class of 3-connected planar graphs. It turned out that our proof that Halin graphs have hamiltonian prisms can be applied to a much larger class of graphs, defined as follows. A *generalized Halin graph (over  $C$ )* is any union of a cycle  $C$  and a tree  $T$  such that  $C$  and  $T$  are edge-disjoint, and  $V(C)$  is the set of all leaves of  $T$ . Thus, compared to the definition of Halin graphs, we do not require the planarity and allow degree 2 vertices in the tree. (See Figure 6 for a drawing of the Petersen graph as a generalized Halin graph.)

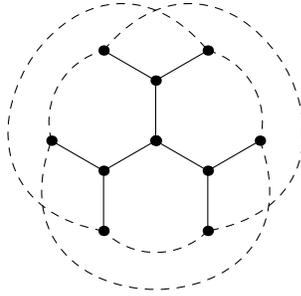


Figure 6: The Petersen graph as a generalized Halin graph over the dashed cycle.

**Lemma 3.3.** *Let  $T$  be a tree and let  $r$  be a vertex of  $T$  of degree at least 2. Then  $T$  contains a spanning system  $\mathcal{P}$  of (possibly trivial) paths, such that*

- (i) *the paths in  $\mathcal{P}$  are vertex-disjoint, and*
- (ii) *each  $P \in \mathcal{P}$  contains exactly one leaf  $v$  of  $T$ , and  $v$  is an endvertex of  $P$ ,*
- (iii)  *$r$  is an endvertex of some path in  $\mathcal{P}$ .*

*Proof.* Orient all edges of  $T$  away from  $r$  to obtain an oriented tree  $\vec{T}$ . Let  $\mathcal{P}$  be a system of directed paths spanning all leaves of  $T$ , satisfying (i) and (ii), and spanning as much of  $T$  as possible. (To see that at least one system with the required properties exists, consider the system of trivial paths  $\{v\}$  for all leaves  $v$  of  $T$ .) Assume  $\mathcal{P}$  does not span  $T$  and choose  $v \notin \bigcup_{P \in \mathcal{P}} V(P)$ . Since  $v$  is not a leaf of  $T$ , there is a vertex  $v^+$  such that  $vv^+ \in E(\vec{T})$ . By a suitable choice of  $v$ , we may assume that  $v^+$  is contained in some path  $P \in \mathcal{P}$ . Since all vertices of  $\vec{T}$  have in-degree  $\leq 1$ , the path  $P$  must begin at  $v^+$ . But then we can augment it by the edge  $vv^+$ , a contradiction with the choice of  $\mathcal{P}$ .  $\square$

**Theorem 3.4.** *Generalized Halin graphs have hamiltonian prisms.*

*Proof.* Let  $G$  be a generalized Halin graph over a cycle  $C$ . If  $C$  is even, then color it alternately blue and yellow, find a system  $\mathcal{P}$  of paths in  $T = G - E(C)$  as in Lemma 3.3 (for an arbitrary  $r \notin V(C)$ ), and color every path from  $\mathcal{P}$  green. This coloring clearly corresponds to a Hamilton cycle of  $G \square K_2$ .

In the rest of the proof, we assume that  $C$  is odd. For each  $e \in E(C)$ , let  $C_e$  be the unique cycle in  $T \cup \{e\}$ . Writing  $|C_e|$  for the length of  $C_e$ , we claim that

$$\sum_{e \in E(C)} |C_e| \text{ is odd.} \quad (2)$$

To begin with,

$$\begin{aligned} \sum_{e \in E(C)} |C_e| &= |C| + \sum_{e \in E(T)} |\{f \in E(C) : e \in C_f\}| \\ &= |C| + \sum_{e \in E(T)} |\{f \in E(C) : f \text{ joins the components of } T - e\}|. \end{aligned}$$

For  $e \in E(T)$ , let  $R_e$  denote the set of all edges of  $C$  joining the two components of  $T - e$ . Since  $|C|$  is odd, it is enough to prove that  $|R_e|$  is even for any  $e \in E(T)$ . Clearly,  $R_e$  is an edge cut in  $G - e$ . It is a standard fact that any cycle intersects any edge cut in an even number of edges. Applying this fact to the cycle  $C$ , we infer that  $R_e$  (which is a subset of  $C$ ) contains an even number of edges, and (2) is established.

From (2) it follows that there is an edge  $h \in E(C)$  with odd  $|C_h|$ . Let  $U$  be the set of vertices of  $C_h$  not incident with  $h$ . Apply Lemma 3.3 to the graph obtained from  $T$  by the contraction of  $U$  to a single vertex  $u$  (discarding loops), setting  $r = u$ . In the resulting system of paths, remove  $u$  from the path which contains it (as an endvertex). The outcome is a system  $\mathcal{P}$  of vertex-disjoint paths in  $G$  spanning  $V(G) - U$ , disjoint from  $U$ , and such that each path has precisely one endvertex on  $C$ .

Make the even cycle  $C' = C \cup C_h - h$  an alternating blue-yellow cycle. Furthermore, color each path in  $\mathcal{P}$  green. As before, this coloring determines a Hamilton cycle in the prism over  $G$ . The proof is finished.  $\square$

## 4 Line graphs

Recall that if  $G = (V, E)$  is a graph, then its *line graph*  $L(G)$  has vertex set  $E$ , and  $e_1, e_2 \in E$  are joined by an edge in  $L(G)$  if  $e_1$  is adjacent to  $e_2$  in  $G$ . We simply say that  $H$  is a line graph if there exists a graph  $G$  such that  $H = L(G)$ .

A prominent conjecture concerning the Hamiltonicity of line graphs was stated by Thomassen [24].



Figure 7: Replacing a vertex  $v$  with two new vertices in the proof of Lemma 4.1.

**Conjecture 4.1 (Thomassen’s conjecture).** *Every 4-connected line graph is hamiltonian.*

The conjecture is open even if we replace ‘4-connected’ by ‘6-connected’. Zhan [28] and Jackson [16] independently proved that 7-connected line graphs are hamiltonian. On the other hand, there are examples of 3-connected non-hamiltonian line graphs: for instance, let  $P'$  be obtained by subdividing each edge of the Petersen graph  $P$  by one vertex. The line graph of  $P'$  is  $P$  with each vertex ‘inflated’ to a triangle. Thus it is 3-connected, and any Hamilton cycle in  $L(P')$  would clearly yield a Hamilton cycle in  $P$ , which does not exist.

In contrast, we show that for prism-hamiltonicity, it is enough if the line graph is 2-connected.

#### 4.1 2-connected line graphs are prism-hamiltonian

In the rest of this section, parallel edges are allowed—so we deal with multigraphs. Most graph definitions carry over naturally to this setting. Multiplicities are counted in vertex degrees (which is important when we speak of *cubic* multigraphs) and in the size of an edge cut (which affects the notion of a *bridgeless* multigraph). In the line graph of a multigraph  $G$ , vertices corresponding to a pair of parallel edges are joined by parallel edges. Thus,  $L(G)$  is, properly speaking, a multigraph too.

In the following lemma, the *contraction* of a subtree  $T \subset G$  consists in contracting every edge of  $T$ , discarding the loops but preserving any parallel edges.

**Lemma 4.1.** *Let  $G$  be a bridgeless multigraph with minimum degree at least 3. Then, there exists a cubic bridgeless multigraph  $G_3$  such that  $G$  can be obtained by the contraction of some pairwise disjoint induced subtrees of  $G_3$ .*

*Proof.* We define the *excess* of  $G$  to be the following sum:

$$\text{exc}(G) = \sum_{v \in V(G)} (\deg_G(v) - 3)$$

The proof proceeds by induction on the excess  $\text{exc}(G)$  of the graph. If  $\text{exc}(G) = 0$ , then  $G$  is cubic and the statement is trivial.

Assume  $\text{exc}(G) > 0$  and consider a vertex  $v$  with  $d := \deg_G(v) > 3$ . Let  $e_1$  be any edge incident with  $v$ . Since  $G$  is bridgeless, there is a cycle of  $G$  which

contains the edge  $e_1$ . Let  $e_2$  be the other edge of this cycle which is incident with  $v$ . Let  $e_3, \dots, e_d$  be the remaining edges incident with the vertex  $v$ . Replace the vertex  $v$  by two new vertices  $v_1$  and  $v_2$ , making the edges  $e_1$  and  $e_d$  incident with  $v_1$ , the remaining edges  $e_2, e_3, \dots, e_{d-1}$  with  $v_2$ , and adding a new edge  $v_1v_2$  (see Figure 7). Let  $G'$  be the resulting graph. Note that  $\text{exc}(G') = \text{exc}(G) - 1$  and  $G$  may be obtained from  $G'$  by contracting the edge  $v_1v_2$ . Furthermore,  $G'$  is bridgeless: the only possible bridge may be  $v_1v_2$ , but this edge is contained in the above cycle through  $e_1$  and  $e_2$ , so  $G'$  is bridgeless indeed. We apply induction to  $G'$ , obtaining a graph  $G_3$ . To get the subtree  $T_v$  of  $G_3$  corresponding to the vertex  $v$  of  $G$ , take the subtrees of  $G_3$  corresponding to  $v_1, v_2 \in V(G')$  and join them by an edge of  $G_3$  incident with both these subtrees. Note that such an edge exists since  $v_1v_2 \in E(G')$ . Clearly,  $T_v$  is an induced subtree of  $G_3$ , because there are no parallel edges between  $v_1$  and  $v_2$  in  $G'$ .  $\square$

We define an *eulerian factor* of a multigraph  $G$  to be a (not necessarily connected) spanning subgraph of  $G$  with all degrees even. Note that the terminology is not quite unified here as different authors might use the term ‘even factor’ or ‘spanning cycle’, reserving ‘eulerian factor’ for a connected spanning subgraph with even degrees.

**Lemma 4.2.** *Let  $G$  be a bridgeless multigraph with minimum degree at least 3. Then,  $G$  contains an eulerian factor  $G'$  such that the degree of each vertex is non-zero in  $G'$ .*

*Proof.* By Lemma 4.1, let  $G_3$  be a cubic bridgeless multigraph such that  $G$  can be obtained from  $G_3$  by the contraction of pairwise disjoint induced subtrees  $\{T_v : v \in V(G)\}$ . Since  $G_3$  is bridgeless and cubic, it contains a 1-factor by the well-known Petersen theorem (which does apply to multigraphs). The complement of the 1-factor is a 2-factor. Let  $E$  be the set of the edges of the 2-factor which correspond to the edges of the original graph  $G$  (i.e., they are not included in any  $T_v$ ). Consider a vertex  $v$ . Since each cycle which enters  $T_v$  has to leave it, the number of edges of the 2-factor incident with  $T_v$  is even. This number is non-zero because  $T_v$  is acyclic. Since the vertices of  $G$  can be obtained from  $G_3$  by the contraction of the subtrees  $T_v$ , each vertex of  $G$  is incident with an even number of edges of  $E$ . Hence  $E$  forms the desired eulerian factor of  $G$ .  $\square$

**Theorem 4.3.** *Let  $G$  be a multigraph. If  $L(G)$  is 2-connected, then the prism over  $L(G)$  contains a Hamilton cycle.*

*Proof.* We first modify  $G$  to get another multigraph  $G_0$ . Let  $E_1$  be the set of edges of  $G$  such that their one of their endvertices has degree one. If  $E_1 = E(G)$ ,  $G$  has to be a star and the statement of the theorem is trivial. Assume henceforth that  $E_1 \neq E(G)$ . Remove the edges in  $E_1$ , along with all the isolated vertices this creates. The resulting graph  $G^-$  is bridgeless, since any bridge would yield a cut-vertex in  $L(G)$ , contradicting the assumption that  $L(G)$  is 2-connected.

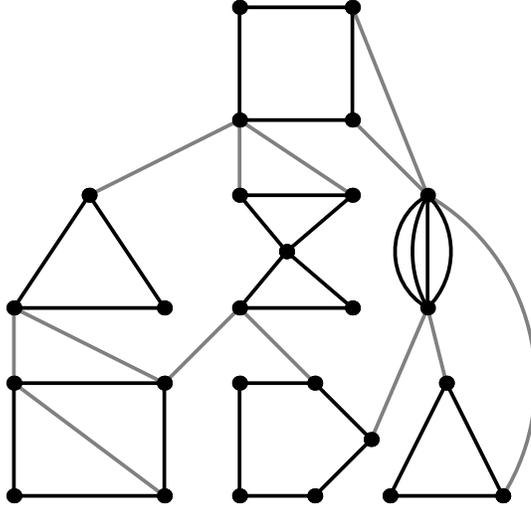


Figure 8: The graph  $G^=$  with the eulerian factor  $F$ . For clarity, some vertices of  $G^=$  are shown with degree 2.

If all the vertices of  $G^-$  have even degrees, then  $G^-$  contains an Euler tour. Hence  $L(G^-)$  is hamiltonian. In fact, it is easy to see that  $L(G)$  is hamiltonian as well, so in particular, its prism has a Hamilton cycle.

In the following, we assume that  $G^-$  contains a vertex of odd degree.

Suppress all the vertices of degree two of  $G^-$ . The resulting multigraph  $G^=$  is bridgeless and its minimum degree is at least 3. Fix an eulerian factor  $F$  of  $G^=$  which exists by Lemma 4.2. Let  $k$  be the number of the components of  $F$ . We shall assign colors to edges of  $G^=$  and later also to those of  $G$ . All the edges included in  $F$  will be colored black (see Figure 8).

Let us introduce two operations we shall use in the proof. A *splitting* of a vertex  $v$  amounts to replacing it with two new vertices  $v'$  and  $v''$  in such a way that each edge incident with  $v$  is made incident with exactly one of  $v'$  and  $v''$ . No edge  $v'v''$  is added. Note that if a multigraph  $H$  is obtained by splitting some vertices of  $G$ , then  $L(H)$  is a spanning subgraph of  $L(G)$ . In particular, to prove that  $L(G)$  is (prism-)hamiltonian, it suffices to prove the same for  $L(H)$ .

The *detachment* of an edge  $uv$  from the vertex  $v$  consists in splitting  $v$  into two vertices such that one of the new vertices is incident only with the edge  $uv$ .

Assume first that the eulerian factor  $F$  is 2-regular, i.e., each of its components is a cycle. (See Figure 9.) The general case will be addressed at the end of the proof. Choose a set of  $k - 1$  edges such that  $F$  together with these edges forms a connected subgraph of  $G^=$ . Color these  $k - 1$  edges red. We claim that  $G^=$  must contain an edge which is neither black nor red. If it does not, then any red edge is a bridge in  $G^=$ , which is assumed not to exist. Thus all edges are black, which implies  $k = 1$ . Hence  $G^=$  is a cycle, which contradicts the assumption that  $G^-$  contains a vertex of odd degree. We have shown that there is some edge  $o$  which

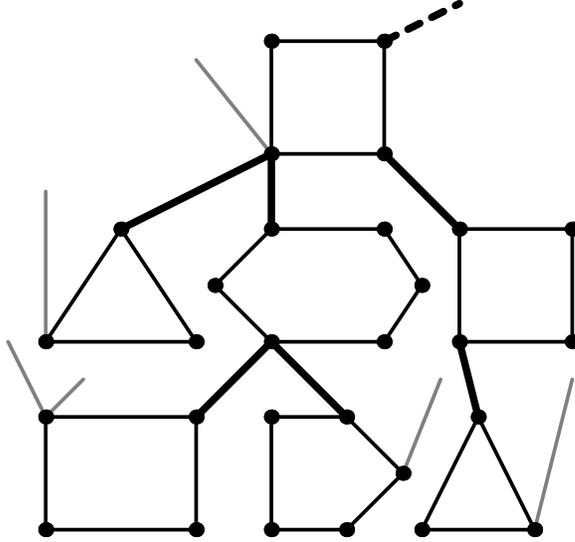


Figure 9: Black, red (drawn as bold), orange (drawn as bold and dashed) and gray edges of  $G^=$ . Again, some degrees are 2 for the sake of clarity.

is neither red nor black as claimed.

Color the edge  $o$  in orange and detach it from one of its endvertices. The edges that have no color so far are now colored gray, and each gray edge is detached from an arbitrary endvertex of it. (Cf. Figure 9.)

We now carry the coloring over to the original graph  $G$ . All the edges of each path comprised of suppressed vertices of degrees 2 get the color of the corresponding edge of  $G^=$  (Figure 10). The removed pendant edges of  $E - E_1$  are colored gray and each of them is detached from one of its end-vertices. Gray edges incident solely with orange (red) edges are recolored orange (red), respectively. Let  $G_0$  be the resulting colored graph (Figure 11).

We use the coloring scheme introduced in Section 2 to prove that the prism over  $L(G_0)$  (and thus also the prism over  $L(G)$ ) contains a Hamilton cycle. The black cycles together with red edges form a tree-like structure. Root this tree at the (unique) cycle incident with the orange edge  $o$ . We form yellow-blue cycles first (stressing that the cycles exist in the line graph, not in  $G_0$  itself). Apply the following to each black cycle of  $G_0$ . Let  $E_C$  be the set of all black edges forming the cycle together with all the gray, red and orange edges incident with it. Let  $e_r$  be the red edge joining the black cycle to the parent cycle, i.e., the cycle closer to the root. If the black cycle is the root cycle, then  $e_r$  is the orange edge incident with it. The yellow-blue cycle is obtained as follows (see Figure 12):

- If  $|E|_C$  is even, the cycle is created on the vertices of  $L(G)$  corresponding to the edges in  $E_C \subseteq E(G_0)$ .
- If  $|E|_C$  is odd, the cycle is created on the vertices of  $L(G)$  corresponding to

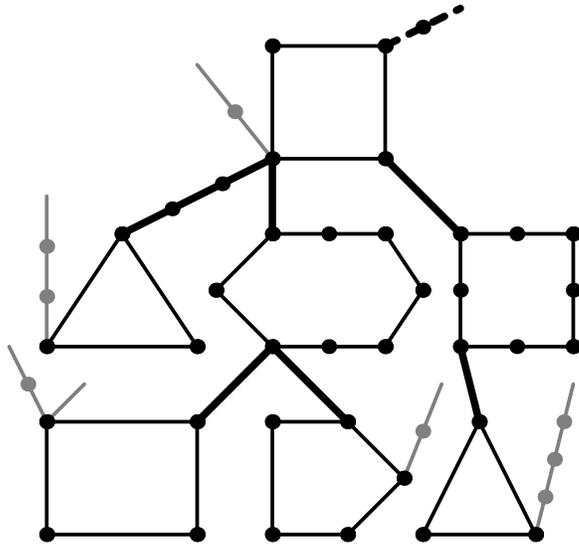


Figure 10: Colors assigned to edges of  $G$  before restoring any pendant edges. The used notation is the same as in Figure 9.

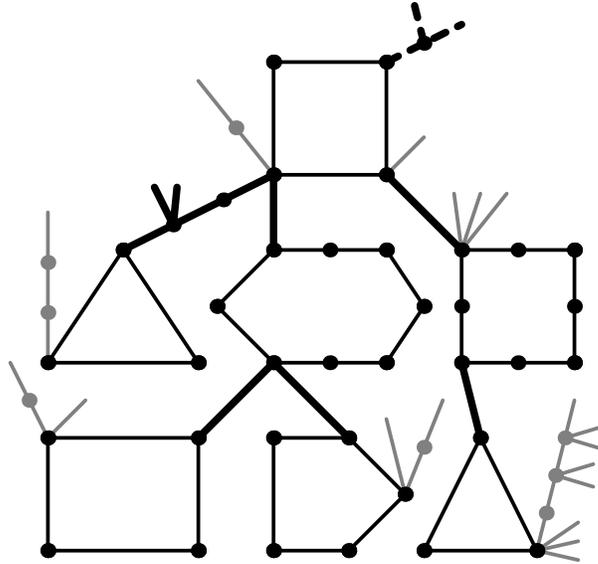


Figure 11: The graph  $G_0$  with all its edges. The used notation is the same as in Figure 9.

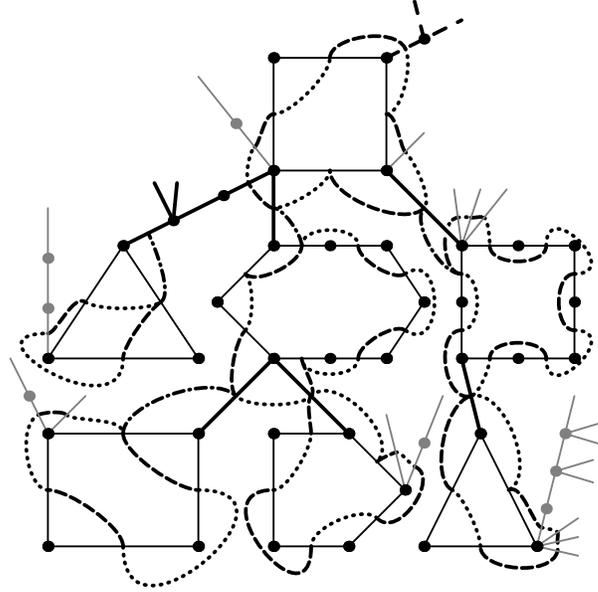


Figure 12: The graph  $G_0$  with the created yellow-blue cycles corresponding to the black cycles and green edges joining red edges not included into the yellow-blue cycles. The used notation is the same as in Figures 3 and 9.

the edges of  $G_0$  in  $E_C - \{e_r\}$ , and the vertex corresponding to  $e_r$  is joined by a green edge to an arbitrary vertex corresponding to a black edge incident with  $e_r$ .

Next, we add green edges (see Figure 13):

- Add green paths corresponding to red paths with suppressed vertices of degree two. These paths include incident red pendant edges if there are any.
- Add green paths corresponding to gray subtrees ending in a gray edge incident with a black cycle. Do the same for the orange subtree ending in the orange edge incident with the root black cycle. These green paths end at the vertex corresponding to their green/orange edges incident with the black cycles.

It is straightforward to check that the yellow-green-blue edges represent a Hamilton cycle in the prism over  $G_0$  through the correspondence explained in Section 2.

It remains to explain how to deal with the case where  $F$  (the eulerian factor of  $G^=$ ) has vertices of degree larger than 2. We make each component of  $F$  into a cycle by splitting all vertices  $w$  of degree  $\geq 4$  in  $F$ ; the edges incident with  $w$  can be made incident with any of the vertices obtained by splitting  $w$  (compare Figs. 8 and 9). As remarked above, it suffices to prove the Hamiltonicity of the

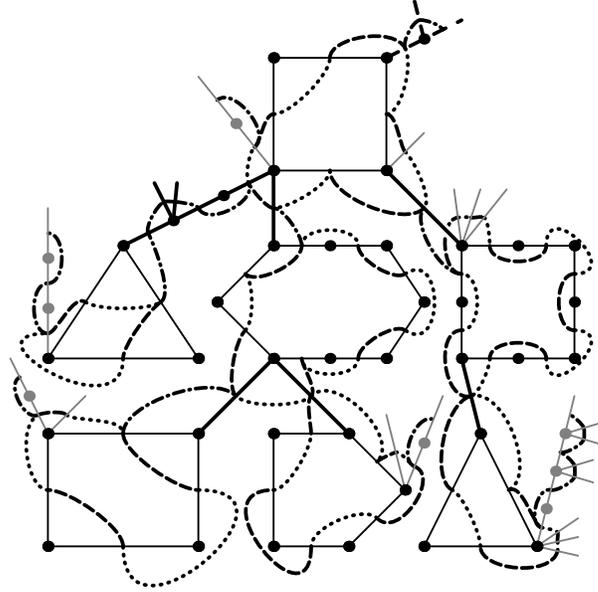


Figure 13: The graph  $G_0$  with all the yellow, green and blue connections. The used notation is the same as in Figures 3 and 9.

prism over the multigraph obtained from the splitting. We have reduced the general situation to the former case.  $\square$

An immediate corollary of Theorem 4.3 is the following:

**Corollary 4.4.** *The prism over the line graph of any bridgeless graph is hamiltonian.*

## 4.2 Claw-free graphs

Many results on line graphs carry over to the wider class of *claw-free graphs*, i.e., graphs containing no induced subgraph isomorphic to  $K_{1,3}$ . For instance, it is known [23] that 7-connected claw-free graphs are hamiltonian, and that Thomassen's conjecture implies that 4-connected claw-free graphs are hamiltonian. We conjecture the following:

**Conjecture 4.2.** *Any 2-connected claw-free graph is prism-hamiltonian.*

In fact, the conjecture has recently been shown to be true by Čada [6]:

**Theorem 4.5.** *Any 2-connected claw-free graph is prism-hamiltonian.*

## 5 4-regular graphs

4-regular, connected graphs clearly have a 2-walk (an Eulerian cycle is a 2-walk). Nash-Williams [20] posed the question whether every 4-connected 4-regular graph is hamiltonian. The first example showing that this is not the case was the graph  $M$  in Figure 14a, constructed by Meredith [5]. The construction is as follows. Let  $P'$  be the (4-regular) graph obtained by doubling a 1-factor in the Petersen graph. For each vertex  $v$  of  $P'$ , take a copy  $H_v$  of the complete bipartite graph  $K_{3,4}$ . The *Meredith graph*  $M$  arises from the disjoint union of these copies by adding edges in such a way that  $M$  is 4-regular, and that there is an edge between  $H_v$  and  $H_w$  whenever  $vw$  is an edge of  $P'$ . Clearly,  $M$  is defined uniquely up to isomorphism. To see that it is non-hamiltonian, observe that any Hamilton cycle would have to enter and leave each  $K_{3,4}$  just once, and so it would determine a Hamilton cycle in the Petersen graph, which does not exist. This construction can be applied to any cubic 3-connected graph to obtain a 4-regular 4-connected graph and if the cubic graph is not hamiltonian neither will the resulting 4-regular graph be. Thus there are even bipartite 4-regular, 4-connected non hamiltonian graphs.

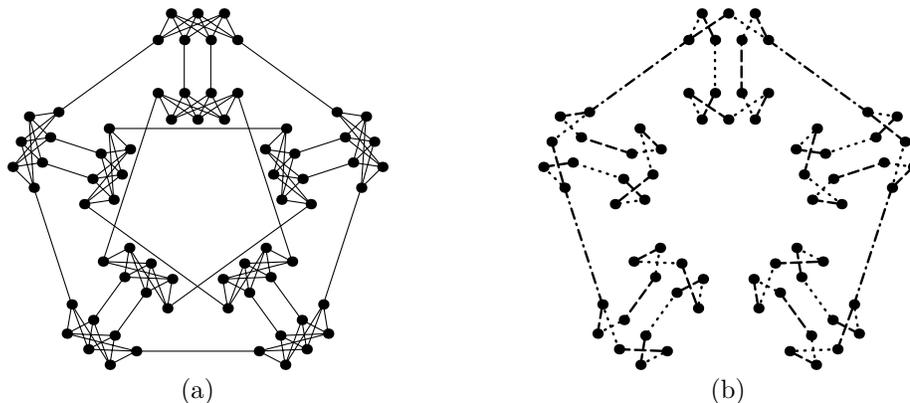


Figure 14: (a) The Meredith graph  $M$ . (b) A spanning even cactus of  $M$ . We use the same notation as in Figure 3.

**Proposition 5.1.** *The Meredith graph  $M$  is prism-hamiltonian.*

*Proof.* By Proposition 2.1, it is sufficient to exhibit a spanning even cactus in  $M$ . One such cactus is shown in Figure 14b.  $\square$

Proposition 5.1, together with the fact that other similarly constructed examples are also prism-hamiltonian, lead us to the following attempt to resuscitate Nash-Williams conjecture.

**Problem 1.** *Are all 4-connected 4-regular graphs prism-hamiltonian?*

An interesting class of 4-regular graphs consists of the line graphs of cubic graphs. By Theorem 4.3, if such a graph is 2-connected, then it has a hamiltonian prism. It follows that if Problem 1 is restricted to line graphs, the answer is affirmative. In connection to this question, it is perhaps worth noting that Conjecture 4.1 is easily seen to be equivalent to the statement that all 4-connected 4-regular *line* graphs are hamiltonian.

## 6 Toughness and connectivity

As mentioned in Section 1, there are examples of graphs of arbitrarily high connectivity which have 2-walks, but whose prisms are non-hamiltonian. We shall now construct one such family of graphs. For a positive integer  $k$ , let  $H_k$  be the graph consisting of three copies  $H_k^1, H_k^2, H_k^3$  of the complete bipartite graph  $K_{2k, 4k-1}$ , and a matching  $M$  connecting one half of the smaller color class of  $H_k^1$  to one half of the smaller color class of  $H_k^2$ , and similarly for the other pairs of indices. (See Figure 15 for a picture of  $H_2$ .) The graph  $H_k$  is  $2k$ -connected, has a 2-walk, and the following argument shows that its prism is not hamiltonian.

Assume that the prism over  $H_k$  has a Hamilton cycle  $C$ . The cycle intersects the prism over  $H_k^1$  in a union  $C^1 = P_1 \cup \dots \cup P_s$  of disjoint paths. We aim to show that  $s = 1$ . Let  $A$  and  $B$  denote the smaller and the larger color class of  $H_k^1$ , respectively. Consider the path  $P_i$ , where  $1 \leq i \leq s$ . Assume  $P_i$  contains  $m$  vertices of  $A \square K_2$ . These vertices split  $P_i$  up into  $m - 1$  paths, each of which contains at most 2 vertices from  $B \square K_2$ . Thus the number of vertices of  $B \square K_2$  on  $P_i$  is at most  $2m - 2$ . Summing over all  $i$ , we get

$$2|B| \leq 4|A| - 2s,$$

since  $P_1 \cup \dots \cup P_s$  spans the prism over  $H_k^1$ . Thus

$$s \leq 2|A| - |B| = 1$$

as claimed. By symmetry,  $C$  must intersect the prisms over  $H_k^2$  and  $H_k^3$  in contiguous paths ( $C^2$  and  $C^3$ , respectively) as well.

Since each  $H_k^j$  has an odd number of vertices, the path  $C^j$  enters  $H_k^j \square K_2$  and leaves it in different copies of  $H_k$ . This, however, cannot be true for all three  $H_k^j$ 's simultaneously. Therefore, the prism over  $H_k$  contains no Hamilton cycle  $C$ .

The above considerations are closely related to the notion of toughness. A graph  $G$  is *k-tough* if the removal of any  $m$  vertices yields a graph with at most  $m/k$  components. The *toughness* of  $G$  is the maximum  $k$  such that  $G$  is *k-tough* (or  $\infty$  if  $G$  is complete). In 1973, Chvátal [8] stated the beautiful conjecture that every 2-tough graph is hamiltonian. Only in 2000, the conjecture was disproved by Bauer, Broersma and Veldman [4] by constructing non-hamiltonian graphs of toughness  $9/4 - \varepsilon$  (for small  $\varepsilon$ ). A weaker form of the conjecture, that there is some  $k$  such that toughness  $k$  implies Hamiltonicity, is still open.

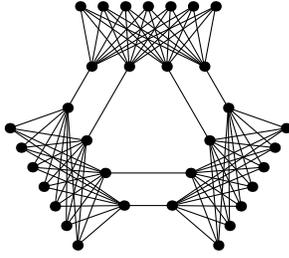


Figure 15: The graph  $H_2$ .

The construction from [4] was modified by Ellingham and Zha [10] who obtained  $(17/24 - \varepsilon)$ -tough graphs with no 2-walk. An upper bound for toughness that guarantees the existence of a 2-walk was also obtained in [10]: every 4-tough graph has a 2-walk. These are the best bounds available, but a conjecture from [17] states that the truth is much closer to the lower bound; namely that a toughness of 1 is sufficient for the existence of a 2-walk. This would improve a result of Win [27] that all 1-tough graphs have 3-trees.

We present another modification of the above construction which gives  $(9/8 - \varepsilon)$ -tough graphs whose prisms are not hamiltonian. Consider the graph  $A$  as in Figure 16 and observe that its prism has no Hamilton path from  $a$  to  $a^*$ .

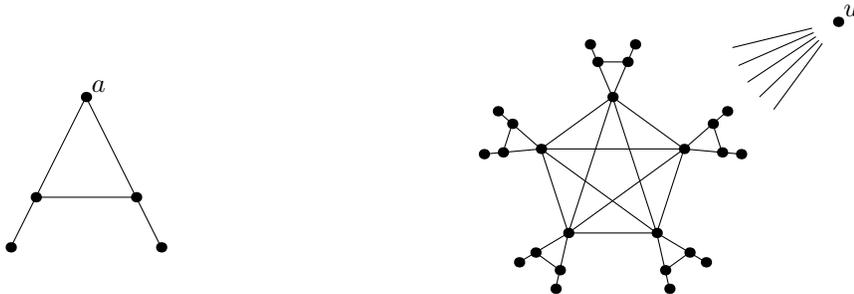


Figure 16: Left: The graph  $A$ . Right: The graph  $G_1$  (the vertex  $u$  is adjacent to all vertices).

Take  $4n+1$  disjoint copies  $A_1, \dots, A_{4n+1}$  of  $A$  and add all edges between copies of the vertex  $a$ . Form a graph  $G_n$  by adding an independent set  $U$  of  $n$  vertices which are adjacent to every vertex outside  $U$ . (See the graph  $G_1$  in Figure 16.)

**Proposition 6.1.** *The prism over the graph  $G_n$  is non-hamiltonian. The toughness of  $G_n$  approaches  $9/8$  as  $n \rightarrow \infty$ .*

*Proof.* This is a straightforward modification of the argument from [4]. Any Hamilton cycle  $C$  in  $G_n \square K_2$  contains  $2n$  vertices equal to  $u$  or  $u^*$  for some  $u \in U$ . Removing these  $2n$  vertices,  $C$  breaks up into at most  $2n$  paths, which

have a total of  $\leq 4n$  endvertices. Since  $G_n$  contains  $4n + 1$  copies of  $A$ , some copy  $A_i$  contains no endvertex of any of the paths. This means that  $C$  covers all of  $A_i$  by a single path, entering at (the copy of)  $a$  and leaving at (the copy of)  $a^*$ . As noted above, this is impossible as  $A \square K_2$  has no Hamilton path from  $a$  to  $a^*$ . Hence  $C$  cannot exist.

We compute the toughness of  $G_n$ . A *toughness set* is any nonempty proper subset  $T$  of  $V(G_n)$  with the smallest possible ratio between  $|T|$  and the number of components of  $G_n - T$ . Clearly, if  $T$  is a toughness set, then  $U \subset T$  and  $T$  contains no vertex whose degree in  $A_i$  is 1. Thus each  $A_i$  has only three candidates for the membership in  $T$ . It is not hard to see that one possible toughness set contains (besides  $U$ ) from each  $A_i$  the two vertices of degree 3 in  $A_i$ . The toughness of  $G_n$  is therefore  $(n + 2(4n + 1))/(1 + 2(4n + 1)) = (9n + 2)/(8n + 3)$  which tends to  $9/8$  as claimed.  $\square$

One can consider an analogue of the weaker conjecture of Chvátal.

**Conjecture 6.1.** *There is a constant  $k$  such that the prism over any  $k$ -tough graph is hamiltonian.*

## 7 The square of a graph

The  $k$ -th power  $G^k$  of a graph  $G$  is the graph on the same vertices as  $G$ , with two distinct vertices  $x, y$  joined by an edge whenever their distance in  $G$  is at most  $k$ . A famous result of Fleischner [11] states that the square  $G^2$  of any 2-connected graph is hamiltonian. For prism-hamiltonicity, we can even relax the assumption of 2-connectivity.

**Theorem 7.1.** *Let  $T$  be a tree with more than one vertex. Then the prism over its square  $T^2$  is hamiltonian.*

*Proof.* We shall show that the prism over  $T^2$  has a particular type of a Hamilton cycle as described below. Let us first introduce a piece of notation. For a 2-factor  $C$  in  $T^2 \square K_2$  and the associated coloring of  $T^2$  as in Section 2, we let  $S(C)$  denote the spanning subgraph of  $T^2$  consisting of all edges which are assigned some color. We claim that the prism over  $T^2$  contains a Hamilton cycle with the following property (\*):

- (i) in the associated coloring of  $T^2$ , green edges are a subset of  $E(T)$ , and each green edge is a cut edge in  $S(C)$ ,
- (ii) adjacent green edges share a vertex whose degree in  $T$  is 2,
- (iii) no leaf of  $T$  has type GG or BBYY.

The proof is by induction on the number of vertices. The statement is trivial if  $T$  is a tree on  $\leq 3$  vertices. Assume next that  $T$  is a star on  $n \geq 4$  vertices with central vertex  $v$ . The square  $T^2$  is then the complete graph  $K_n$ . If  $n$  is even, take any Hamilton cycle in  $T^2$  and color it blue-yellow to obtain the coloring defining  $C$ . If  $n$  is odd, take an even cycle in  $T^2$  through all the leaves of  $T$ , color it blue-yellow and add a green edge from any of the leaves to  $v$ . Clearly, this coloring has the required properties.

If  $T$  is not a star, then let  $v$  be a vertex all of whose neighbors are leaves of  $T$ , except for exactly one vertex  $w$ . Let  $L$  be the set of leaves adjacent to  $v$ . By induction, there is a Hamilton cycle  $C'$  in the prism over  $(T - L)^2$  satisfying (\*). To extend the associated coloring  $c'$  to  $T^2$ , we distinguish two cases based on the type of  $v$  in  $S(C')$ .

*Case 1.* The type of  $v$  is BY (see Figure 17). Noting that the subtree  $T_1 \subset T$  on the vertex set  $L \cup \{v\}$  is a star, find a coloring  $c_1$  of  $T_1^2$  as described above. The extended coloring is obtained simply as the union (superposition) of  $c_1$  and  $c'$ . It is straightforward to check that the extension determines a Hamilton cycle in the prism over  $T^2$  and preserves property (\*). We omit the details.

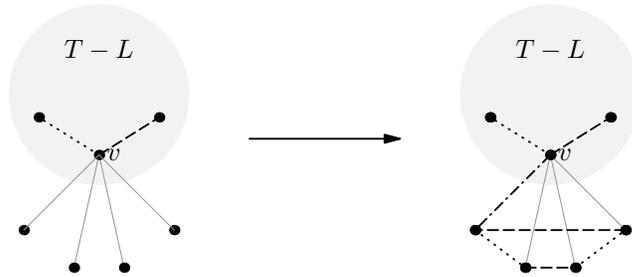


Figure 17: Extending the coloring when  $v$  is of type BY (Case 1). The notation is the same as in Figure 3.

*Case 2.* The type of  $v$  is G or BYG (see Figure 18). This time, consider the star  $T_2$  on  $L \cup \{v, w\}$ . Find a coloring  $c_2$  of  $T_2^2$  as above, choosing the green edge (if there is one) to be different from  $vw$ . By (i), the green edge of  $S(C')$  adjacent to  $v$  is  $vw$  and its removal disconnects  $S(C')$ . The desired coloring  $c$  is obtained from  $c'$  by first uncoloring  $vw$  and then adding the coloring  $c_2$ .

We need to show that the associated 2-factor  $C_2$  (in the prism) is a Hamilton cycle. Let  $P$  be the path in  $C_2$  between  $v$  and  $w$ , and let  $Q$  be the path in  $C_2$  between  $v^*$  and  $w^*$ . Note that by the construction,  $P$  and  $Q$  are disjoint, and all their internal vertices are in  $L \square K_2$ . Furthermore, it is not hard to see that replacing each of  $P$  and  $Q$  by an edge, we obtain  $C'$ . The claim that  $C_2$  is a Hamilton cycle follows. Again, it is easy to check that (\*) is preserved.  $\square$

Theorem 7.1 directly implies the following corollary.

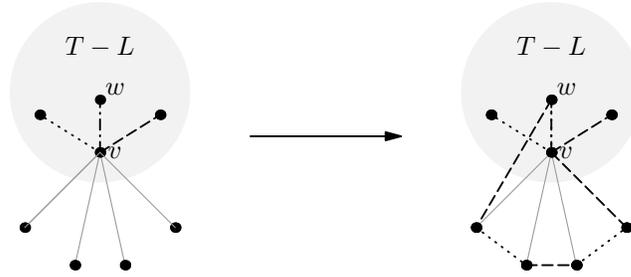


Figure 18: Extending the coloring when  $v$  is of type BYG (Case 2). The notation is the same as in Figure 3.

**Corollary 7.2.** *The square of any connected graph is prism-hamiltonian.*  $\square$

As demonstrated in this paper, many questions may be asked concerning prism Hamiltonicity. Let us conclude with a question which does not quite fit into any of the previous sections: Is there an analogue of the well-known Bondy-Chvátal closure concept for prism-hamiltonicity? In particular, is the following true?

**Problem 2.** *Let  $G$  be a graph of order  $n$  and let  $x$  and  $y$  be two non-adjacent vertices such that the sum of their degrees is at least  $n$ . Is it true that  $G$  has a hamiltonian prism if and only if  $G + xy$  does?*

The answer to this question is negative but the statement becomes true when the constraint on the sum of degrees is replaced by  $4n/3 - 4/3$  as shown by the second author and Stacho [19]:

## References

- [1] B. Alspach and M. Rosenfeld, “On Hamilton decompositions of prisms over simple 3-polytopes”, *Graphs Comb.* **2** (1986) 1–8.
- [2] D. Barnette, “Trees in polyhedral graphs”, *Canad. J. Math.* **18** (1966) 731–736.
- [3] D. P. Biebighauser and M. N. Ellingham, “Prism-hamiltonicity of triangulations”, manuscript, 2005.
- [4] D. Bauer, H. J. Broersma and H. J. Veldman, “Not every 2-tough graph is hamiltonian”, *Discrete Appl. Math.* **99** (2000) 317–321.
- [5] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications* (Macmillan, London, and Elsevier, New York, 1976).

- [6] R. Čada, “2-connected claw-free graphs are prism-hamiltonian”, submitted.
- [7] R. Čada, T. Kaiser, M. Rosenfeld and Z. Ryjáček, “Hamiltonian decompositions of prisms over cubic graphs”, *Discrete Math.*, to appear.
- [8] V. Chvátal, “Tough graphs and hamiltonian circuits”, *Discrete Math.* **5** (1973) 215–228.
- [9] M. N. Ellingham, “Spanning paths, cycles, trees and walks for graphs on surfaces”, *Congr. Numerantium* **115** (1996) 55–90.
- [10] M. N. Ellingham and X. Zha, “Toughness, trees, and walks”, *J. Graph Theory* **33** (2000) 125–137.
- [11] H. Fleischner, “The square of every two-connected graph is hamiltonian”, *J. Comb. Theory Ser. B* **16** (1974) 29–34.
- [12] H. Fleischner, “The prism of a 2-connected, planar, cubic graph is hamiltonian (a proof independent of the four colour theorem)”, in *Graph theory in memory of G. A. Dirac*, Volume 41 of *Ann. Discrete Math.*, 1989), 141–170.
- [13] Z. Gao and R. B. Richter, “2-walks in circuit graphs”, *J. Comb. Theory Ser. B* **62** (1994) 259–267.
- [14] R. J. Gould, Advances on the hamiltonian problem—a survey. *Graphs Comb.* **19** (2003), 7–52.
- [15] B. Grünbaum, Polytopes, graphs, and complexes, *Bull. AMS* **76** (1970), 1131–1201.
- [16] B. Jackson, “Hamilton cycles in 7-connected line graphs”, unpublished manuscript, 1989.
- [17] B. Jackson and N. C. Wormald, “ $k$ -walks of graphs”, *Australas. J. Combin.* **2** (1990) 135–146.
- [18] J. R. Johnson, “Long cycles in the middle two layers of the discrete cube”, *J. Comb. Theory Ser. A* **105** (2004) 255–271.
- [19] D. Král’, L. Stacho: “Closure for the property of having a hamiltonian prism”, submitted.
- [20] C. St. J. A. Nash-Williams, “Hamiltonian arcs and circuits”, in *Recent trends in graph theory*, Volume 185 of *Lecture Notes in Math.*, (Springer, Berlin, 1971) 197–210.
- [21] P. Paulraja, “A characterization of hamiltonian prisms”, *J. Graph Theory* **17** (1993) 161–171.

- [22] M. Rosenfeld and D. Barnette, “Hamiltonian circuits in certain prisms”, *Discrete Math.* **5** (1973) 389–394.
- [23] Z. Ryjáček, “On a closure concept in claw-free graphs”, *J. Comb. Theory Ser. B* **70** (2) (1997) 217–224.
- [24] C. Thomassen, “Reflections on graph theory”, *J. Graph Theory* **10** (1986) 309–324.
- [25] M. Tkáč and H.-J. Voss, “On  $k$ -trestles in chordal polyhedral graphs”, preprint MATH-AL-15-2002, Technische Universität Dresden, 2002.
- [26] W. T. Tutte, “A theorem on planar graphs”, *Trans. Amer. Math. Soc.* **82** (1956) 99–116.
- [27] S. Win, “On a connection between the existence of  $k$ -trees and the toughness of a graph”, *Graphs Comb.* **5** (2) (1989) 201–205.
- [28] S. Zhan, “On hamiltonian line graphs and connectivity”, *Discrete Math.* **89** (1991) 89–95.