

A Dirac theorem for trestles*

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Abstract

A k -subtrestle in a graph G is a 2-connected subgraph of G of maximum degree at most k . We prove a lower bound on the order of a largest k -subtrestle of G , in terms of k and the minimum degree of G . A corollary of our result is that every 2-connected graph with n vertices and minimum degree at least $2n/(k+2)$ contains a spanning k -subtrestle. This corollary is an extension of Dirac's Theorem.

1 Introduction

One of the basic results of Graph Theory is Dirac's minimum degree condition for the hamiltonicity of a graph [4] (see also [2, Theorem 2.1]):

Theorem 1 (Dirac's Theorem). *Every graph with $n \geq 3$ vertices and minimum degree at least $n/2$ contains a Hamilton cycle.*

Dirac [4] derived the following corollary of Theorem 1, given as Theorem 2.15 in [2]:

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Theorem 2 (Dirac). *Every 2-connected graph with n vertices and minimum degree δ contains a cycle of length at least $\min\{2\delta, n\}$.*

We will be concerned with an extension of the above results from cycles to structures known as (sub)trestles. Given a positive integer k , a k -subtrestle in a graph G is a 2-connected subgraph $H \subset G$ with maximum degree $\Delta(H) \leq k$. A k -trestle in G is a spanning k -subtrestle. Thus, a 2-trestle is exactly a Hamilton cycle. This concept was first studied (with different terminology) by Barnette [1] for 3-connected planar graphs. Further results on the existence of k -trestles in embedded graphs can be found in [7, 8, 9, 10].

In the present note, we will establish an extension of Theorem 2 to trestles:

Theorem 3. *Let G be a 2-connected graph with n vertices and minimum degree δ and let $k \geq 2$. Then G contains a k -subtrestle H with*

$$|V(H)| \geq \min \left\{ \left\lfloor \frac{\delta(k+2)}{2} \right\rfloor, n \right\}. \quad (1)$$

Observe that Theorem 2 follows from Theorem 3 by setting $k = 2$. Another corollary, in the direction of Theorem 1, was originally conjectured by M. Tkáč:

Corollary 4. *Every 2-connected graph with n vertices and minimum degree at least $2n/(k+2)$ contains a k -trestle.*

To see that Theorem 3 implies Corollary 4, note that the degree condition in Corollary 4 implies that the minimum on the right hand side of (1) equals n . Thus, by Theorem 3, G contains a k -trestle.

The following family of examples shows that Theorem 3 is optimal. Let $a, k \geq 2$ be integers. Set $b = \lfloor ak/2 \rfloor + 1$ and consider the complete bipartite graph $K_{a,b}$. Note that the minimum degree of $K_{a,b}$ is a . Using Theorem 3, it is easy to see that $K_{a,b}$ contains a k -subtrestle on at least $a + b - 1$ vertices. To demonstrate the optimality of Theorem 3, we will show that $K_{a,b}$ does not contain a k -subtrestle on $a + b$ vertices — equivalently, a k -trestle.

Suppose $K_{a,b}$ contains a k -trestle H . Then the number of edges of H (say, m) is at most ak by the maximum degree condition for H . On the other hand, since every vertex of $K_{a,b}$ has degree at least 2 in H , $m \geq 2b$. Combining the inequalities, we find

$$\frac{ak}{2} - \left\lfloor \frac{ak}{2} \right\rfloor \geq 1,$$

a contradiction. Thus, $K_{a,b}$ contains no k -trestle and the optimality of Theorem 3 is established. Furthermore, this example also shows that Corollary 4 is optimal.

In the rest of this section, we fix the necessary terminology. The reader is referred to standard graph theory textbooks such as [3] for any undefined notions and notation.

All the graphs will be undirected and simple. Let H be a subgraph of a graph G . If v is a vertex of G , then $N_H(v)$ denotes the set of neighbors of v in H , and we write $d_H(v)$ for $|N_H(v)|$. We let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of G , respectively.

For vertices u, v of G , a uv -path is a path whose endvertices are u and v . A uH -path is a path whose endvertices are u and a vertex of H , and none of whose internal vertices is contained in H . As usual in the theory of ear decompositions, we define an H -ear to be a path of length at least 2 in G with its endvertices in H and otherwise disjoint from H . Two paths are *internally vertex-disjoint* if each vertex in their intersection is an endvertex of both of the paths.

If K is a component of $G - V(H)$, then \tilde{K} denotes the subgraph of G obtained from K by adding all the edges of G with one endvertex in K and the other in H , together with all such endvertices.

If u and v are vertices on a path P , we write uPv for the part of P between u and v inclusive. The concatenation of paths uPv and vQw (which in general may not be a path) is denoted by $uPvQw$. Similarly, for the concatenation of an edge uv and a path vQw , we write $uvQw$, etc.

2 Preliminary observations

We will need a lemma on long paths in graphs where almost all graphs have high degree. The lemma follows directly from [5, Proposition 1], but it also has a simpler proof which we give below. (See also [6], where a more general result is claimed without a proof.)

Lemma 5. *Let G be a 2-connected graph on at least three vertices, and assume that every vertex of G , except possibly for vertices u and v , has degree at least δ . Then G contains a uv -path P of length at least δ .*

Proof. Let $p = \min \{d_G(u), d_G(v)\}$. We define a graph G^* by taking $k \geq \delta/p$ copies of G and identifying all the copies of the vertex u ; we also identify all the copies of v . (If uv is an edge of G , we only include it once.) Note that G^* is 2-connected and $\delta(G^*) \geq \delta$. By Theorem 2, it contains a cycle C which is either hamiltonian or has length at least 2δ .

If C is hamiltonian, then the part of C in any copy of G gives a Hamilton uv -path in G . Since G contains vertices of degree δ , this path must have length at least δ and we are done.

Thus, we may assume that the length of C is at least 2δ . If all of C resides in one copy of G , then let P_u and P_v be disjoint (possibly trivial) paths from C to u and v , respectively. Choosing P to be a subpath of C of length at least δ , we observe that the concatenation of P_u , P and P_v is a uv -path of length at least δ .

The only remaining case is that the length of C is at least 2δ and C is contained in the union of two copies of G in G^* . Then the part of C in one of the two copies

must have length at least δ . □

The following lemma is our main technical device.

Lemma 6. *Let H be a subgraph of a 2-connected graph G such that $|V(H)| \geq 2$. Assume that the minimum degree of G is at least $\delta \geq 3$. For each component K of $G - V(H)$, at least one of the following conditions holds:*

- (a) K is trivial (that is, $|V(K)| = 1$),
- (b) K contains at least two vertices whose degree in K is at most $\delta/2 - 1$, or
- (c) \tilde{K} contains an H -ear of length at least $(\delta + 3)/2$.

Proof. Suppose that K is a component of $G - V(H)$ satisfying neither (a) nor (b). We prove that K has property (c). By the assumption, K has at least two vertices. Furthermore, if $|V(K)| = 2$, then since (b) does not hold, we must have $\delta = 3$, in which case (c) is satisfied. Thus, we may assume that K has at least 3 vertices.

If K is 2-connected, then we can use Lemma 5. By the 2-connectedness of G , there are vertices $u_1, u_2 \in V(K)$ such that each u_i has a neighbor in H and the neighbors are distinct. Clearly, we may choose u_1 as the vertex of degree at most $\delta/2 - 1$ in K if it exists. By Lemma 5, K contains a u_1u_2 -path P of length at least $(\delta - 1)/2$. Adding the edges joining each u_i to its neighbor in H , we obtain an H -ear of length $(\delta + 3)/2$ in \tilde{K} . Thus, we may suppose that K contains a cut-vertex.

If K contains at least two vertices (say, v_1 and v_2) of degree 1 in K , then $\delta \leq 3$ for otherwise K would satisfy (b). By the 2-connectedness of G , it is easy to find an H -ear of length 3 in \tilde{K} as required.

By the above, we may assume K to have a cut-vertex and contain only at most one vertex of degree 1 in K . Since K has at least two end-blocks and only one can be of order 2, we can choose an end-block B with $|V(B)| \geq 3$. Let b be the cut-vertex incident with B . We define K_B as the graph obtained from K by removing all vertices of B except b .

Since condition (b) is not satisfied, there is at most one vertex in $V(B) - \{b\}$ whose degree in B is at most $\delta/2 - 1$. We distinguish two cases based on whether such a vertex exists or not.

Case 1: *A vertex u of $B - \{b\}$ satisfies $d_B(u) \leq \delta/2 - 1$.* Since all vertices of B other than b and u have degree at least $(\delta - 1)/2$, Lemma 5 implies that B contains a bu -path P of length at least $(\delta - 1)/2$.

Let w be a neighbor of b in K_B . Since G is 2-connected, $G - \{b\}$ contains a wH -path W ; let us denote its endvertex in H by w' . Observe that W is vertex-disjoint from P . We wish to extend the path $P' := w'WwbPu$ to make it an H -ear. To do so, note that since $d_B(u) \leq \delta/2 - 1$, we have

$$d_H(u) \geq \frac{\delta}{2} + 1 \geq \frac{5}{2}.$$

The vertex u thus has at least three neighbors in H , so we can choose one, say u' , which is distinct from w' . Adding the edge uu' to the above path P' , we obtain an H -ear of length at least $(\delta + 5)/2$.

Case 2: All vertices v of $B - \{b\}$ satisfy $d_B(v) \geq (\delta - 1)/2$. Let w_1 be a neighbor of b in K_B , and let w_2 be a neighbor of b in B . Since G is 2-connected, $G - b$ contains a w_1H -path W_1 and a w_2H -path W_2 . These paths are internally vertex-disjoint as b is a cut-vertex of K . For $i = 1, 2$, let w'_i be the endvertex of W_i in H . Furthermore, let u_2 be the neighbor of w'_2 on W_2 . Since all vertices of B except b have degree at least $(\delta - 1)/2$ in B , and since $|V(B)| \geq 3$, Lemma 5 implies that B contains a bu_2 -path Q of length at least $(\delta - 1)/2$.

If $w'_1 \neq w'_2$, then the path $w'_1W_1w_1bQu_2w'_2$ is an H -ear of length at least $(\delta + 5)/2$ in \tilde{K} , and we are done. Thus, we may assume that for all neighbors v of b in G , all vH -paths end in w'_1 . This means that w'_1 is a cutvertex of G separating b from the (nonempty) subgraph $H - w'_1$, a contradiction with the assumption that G is 2-connected. \square

3 Proof of Theorem 3

Let H be a 2-connected subgraph of a graph G and d a positive integer. A d -extension of H is any graph H' which can be obtained as

$$H' = H \cup P_1 \cup \dots \cup P_m,$$

where for each $i = 1, \dots, m$, P_i is an $(H \cup P_1 \cup \dots \cup P_{i-1})$ -ear such that each of its endvertices is an endvertex of at most $d - 1$ of the paths P_1, \dots, P_{i-1} , and P_i is as long as possible with this property. Note that any d -extension of a k -subtrestle of G is a $(k + d)$ -subtrestle of G .

A d -extension H' of H is *maximal* if it has the maximum possible number of vertices among d -extensions of H .

For any d -extension H' of H (maximal or not), the sequence (P_1, \dots, P_m) is called an *ear sequence* of H' (with respect to H). In general, it is not uniquely determined. If the ear sequence is fixed, then for $0 \leq i \leq m$ we write

$$H_i = H \cup P_1 \cup \dots \cup P_i$$

(in particular, $H_0 = H$ and $H_m = H'$).

We call a path *long* if its length is at least $(\delta + 3)/2$, *short* if its length is 2, and *intermediate* otherwise.

Lemma 7. *Let (P_1, \dots, P_m) be an ear sequence of a d -extension H' of a 2-connected subgraph H of G . Then the following holds:*

- (i) *the lengths of the paths P_1, \dots, P_m are non-increasing;*

- (ii) if there is a long H' -ear P and H' is a maximal d -extension of H , then some endvertex of P is an endvertex of d long paths in (P_1, \dots, P_m) ;
- (iii) if there is an intermediate H' -ear P and H' is a maximal d -extension of H , then some endvertex of P is an endvertex of d long or intermediate paths in (P_1, \dots, P_m) .

Proof. (i) Suppose that $1 \leq i < j \leq m$ and P_i is shorter than P_j . Let u_1 and u_2 be the endvertices of P_j . Being 2-connected, H_{j-1} contains a u_1H_{i-1} -path Q_1 and a u_2H_{i-1} -path Q_2 such that Q_1 and Q_2 are vertex-disjoint. For $t = 1, 2$, let v_t be the endvertex of Q_t other than u_t .

Since no internal vertex of P_j is contained in H_{j-1} , the concatenation of Q_1 , P_j and Q_2 is a path. In fact, it is an H_{i-1} -ear, and it is longer than P_i . By the definition of a d -extension, and by symmetry, v_1 is an endvertex of at least d of the paths P_1, \dots, P_{i-1} . Thus, it is not contained in any of the paths P_i, \dots, P_j , and hence all its neighbors in H_j are contained in H_{i-1} . Considering the neighbor of v_1 on Q_1 , we get a contradiction with the choice of Q_1 .

(ii) By (i), there is some ℓ ($1 \leq \ell \leq m$) such that the long paths in (P_1, \dots, P_m) are P_1, \dots, P_ℓ . Suppose that each endvertex of P is an endvertex of fewer than d of these paths. By the maximality of H' , $m > \ell$ for otherwise we could get a larger d -extension of H by adding P to H_ℓ . Similarly as in part (i), we can extend P to a long H_ℓ -ear, none of whose endvertices is an endvertex of more than $d - 1$ of the paths P_1, \dots, P_ℓ . This contradicts the choice of the (intermediate or short) H_ℓ -ear $P_{\ell+1}$. The proof of part (iii) is similar. \square

Let us point out once more that by Lemma 7(i), if an ear sequence of H' contains any long paths, then they precede all the other paths, and short paths appear at the end of the sequence.

We are now ready to prove our main result, Theorem 3.

Proof of Theorem 3. We proceed by induction on k , using $k = 2$ and $k = 3$ as base cases.

Case 1: $k = 2$. In this case, the assertion is true by Theorem 2.

Case 2: $k = 3$. Let H be a cycle in G of length at least 2δ , which exists by Theorem 2. We need to extend H to either a 3-trestle of G , or a 3-subtrestle of G whose order exceeds the order of H by at least

$$\left\lfloor \frac{5\delta}{2} \right\rfloor - 2\delta = \left\lfloor \frac{\delta}{2} \right\rfloor.$$

Thus, let H' be a maximal 1-extension of H and let (P_1, \dots, P_m) be an ear sequence of H' . Set $x = |V(H')| - |V(H)|$. We may suppose that H' is not spanning and $x < \lfloor \delta/2 \rfloor$, for otherwise we are done.

Let m_1, m_2 and m_3 be the number of long, intermediate and short paths among (P_1, \dots, P_m) . Since $x < \delta/2$, we have $m_1 = 0$. By the 2-connectedness of H and

the fact that H' is not spanning, there is a longest H' -ear P . By Lemma 7(ii), P is not long. We distinguish two cases.

Suppose first that P is intermediate and let K be the component of $G - V(H')$ containing the internal vertices of P . By Lemma 6, K contains vertices u_1, u_2 whose degree in K is at most $\delta/2 - 1$. Thus, for $i = 1, 2$, $d_{H'}(u_i) \geq \delta/2 + 1$. Moreover, any two distinct vertices $v_1 \in N_{H'}(u_1)$ and $v_2 \in N_{H'}(u_2)$ are the endvertices of an intermediate H' -ear. By the maximality of H' and Lemma 7(iii), it may be assumed that each vertex in $N_{H'}(u_1)$ except at most one is an endvertex of an intermediate path P_j ($1 \leq j \leq m_2$). Consequently,

$$2m_2 \geq \delta/2.$$

From another point of view, this inequality means that the paths P_1, \dots, P_{m_2} contain at least $\delta/2$ internal vertices, which implies that $x \geq \delta/2$ contrary to our assumption.

Thus, we may assume that P is short. Let v be the internal vertex of P . By Lemma 6, there are at least δ neighbors of v in H' and any two of them determine a short H' -ear. By the maximality of H' , each of these neighbors, except at most one, must be an endvertex of one of the paths P_1, \dots, P_m . Hence

$$2m \geq \delta - 1$$

and we find that $m \geq \lfloor \delta/2 \rfloor$ for each possible parity of δ . Since $x \geq m$, this contradicts our assumption.

Case 3: $k \geq 4$. Using the induction hypothesis, we find a $(k - 2)$ -subtrestle H of G of order at least $\min\{\lfloor \delta k/2 \rfloor, n\}$. The argument is similar to that in Case 2, except we now look for a 2-extension in place of a 1-extension, and the 2-extension should add about double the amount of new vertices compared to the requirement in Case 2.

Let H' be a maximal 2-extension of H and let $x = |V(H')| - |V(H)|$. If $x \geq \delta$, then the order of the k -subtrestle H' is at least

$$\left\lfloor \frac{\delta k}{2} \right\rfloor + \delta = \left\lfloor \frac{\delta(k+2)}{2} \right\rfloor$$

as required. We may thus assume that $x \leq \delta - 1$. Similarly, it may be assumed that H' is not spanning.

As in Case 2, let (P_1, \dots, P_m) be an ear sequence of H' and let m_1, m_2 and m_3 be the number of long, intermediate and short paths in the sequence, respectively. Since

$$x \geq \left\lceil \frac{\delta + 1}{2} \right\rceil \cdot m_1 + 2m_2 + m_3, \quad (2)$$

we have $m_1 \leq 1$ and $m_1 + m_2 \leq (\delta - 1)/2$.

Let P be a longest H' -ear. By Lemma 7(ii) and the fact that $m_1 \leq 1$, P is not long.

Suppose that P is an intermediate path with its internal vertices contained in a component K of $G - V(H')$. Similarly to Case 2, we find that K contains vertices u_1, u_2 , each with at least $\delta/2 + 1$ neighbors in H' . Since none of the ears determined by these vertices can be added to H' to produce a larger 2-extension of H , we may assume that each vertex in $N_{H'}(u_1)$ but one is an endvertex of two paths among $P_1, \dots, P_{m_1+m_2}$. Hence

$$2(m_1 + m_2) \geq 2 \cdot \frac{\delta}{2},$$

and it follows from (2) that $x \geq \delta$, in contradiction with our assumption.

Thus, P must be a short path. We let v be its internal vertex. By the maximality of H' , all neighbors of v except at most one are endvertices of two paths among P_1, \dots, P_m . It follows that

$$2(m_1 + m_2 + m_3) \geq 2d_G(v) - 2. \quad (3)$$

Dividing by 2 and subtracting from inequality (2), we find that

$$x - d_G(v) + 1 \geq \left\lceil \frac{\delta - 1}{2} \right\rceil \cdot m_1 + m_2.$$

The left hand side is non-positive since $x < \delta$ and $d_G(v) \geq \delta$. It follows that $m_1 = m_2 = 0$. By (2) and (3),

$$\delta > x \geq m_3 \geq d_G(v) - 1 \geq \delta - 1$$

and thus $m = m_3 = \delta - 1$ and $d_G(v) = \delta$.

Let W be the set of vertices of H' which are endvertices of two of the paths $P_1, \dots, P_{\delta-1}$. By the above, W contains at least $\delta - 1$ neighbors of v . Since there are exactly $\delta - 1$ paths, we obtain that $|W| = \delta - 1$, $W \subsetneq N_G(v)$ and all endvertices of the paths P_i ($i = 1, \dots, \delta - 1$) are in W . This in particular implies that $\delta \geq 3$.

Let the vertices of the path P_1 be denoted by a, b, c in this order (thus, $a, c \in W$) and let b' be a neighbor of b outside W . Furthermore, let w be the (unique) neighbor of v not in W . If we replace P_1 in the ear sequence of H' by the paths abb' and cvw , we obtain a corresponding d -extension of H which is one vertex larger than H' , a contradiction with the maximality of H' . \square

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