

## PAIRS OF HEAVY SUBGRAPHS FOR HAMILTONICITY OF 2-CONNECTED GRAPHS\*

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**Abstract.** Let  $G$  be a graph on  $n$  vertices. An induced subgraph  $H$  of  $G$  is called heavy if there exist two nonadjacent vertices in  $H$  with degree sum at least  $n$  in  $G$ . We say that  $G$  is  $H$ -heavy if every induced subgraph of  $G$  isomorphic to  $H$  is heavy. For a family  $\mathcal{H}$  of graphs,  $G$  is called  $\mathcal{H}$ -heavy if  $G$  is  $H$ -heavy for every  $H \in \mathcal{H}$ . In this paper we characterize all connected graphs  $R$  and  $S$  other than  $P_3$  (the path on three vertices) such that every 2-connected  $\{R, S\}$ -heavy graph is Hamiltonian. This extends several previous results on forbidden subgraph conditions for Hamiltonian graphs.

**Key words.** forbidden subgraph, heavy subgraph, Hamilton cycle

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**1. Introduction.** We use Bondy and Murty [2] for terminology and notation not defined here and consider finite simple graphs only.

Let  $G$  be a graph. For a vertex  $v \in V(G)$  and a subgraph  $H$  of  $G$ , we use  $N_H(v)$  to denote the set, and  $d_H(v)$  the number, of neighbors of  $v$  in  $H$ . We call  $d_H(v)$  the *degree* of  $v$  in  $H$ . For  $x, y \in V(G)$ , an  $(x, y)$ -*path* is a path  $P$  connecting  $x$  and  $y$ ; the vertex  $x$  will be called the *origin* and  $y$  the *terminus* of  $P$ . If  $x, y \in V(H)$ , the *distance* between  $x$  and  $y$  in  $H$ , denoted  $d_H(x, y)$ , is the length of a shortest  $(x, y)$ -path in  $H$ . When no confusion occurs, we will denote  $N_G(v)$ ,  $d_G(v)$ , and  $d_G(x, y)$  by  $N(v)$ ,  $d(v)$ , and  $d(x, y)$ , respectively.

Let  $G$  be a graph on  $n$  vertices. If a subgraph  $G'$  of  $G$  contains all edges  $xy \in E(G)$  with  $x, y \in V(G')$ , then  $G'$  is called an *induced subgraph* of  $G$ . For a given graph  $H$ , we say that  $G$  is  $H$ -*free* if  $G$  does not contain an induced subgraph isomorphic to  $H$ . For a family  $\mathcal{H}$  of graphs,  $G$  is called  $\mathcal{H}$ -*free* if  $G$  is  $H$ -free for every  $H \in \mathcal{H}$ . If  $H$  is an induced subgraph of  $G$ , we say that  $H$  is *heavy* if there are two nonadjacent vertices in  $V(H)$  with degree sum at least  $n$  in  $G$ . The graph  $G$  is called  $H$ -*heavy* if every induced subgraph of  $G$  isomorphic to  $H$  is heavy. For a family  $\mathcal{H}$  of graphs,  $G$  is called  $\mathcal{H}$ -*heavy* if  $G$  is  $H$ -heavy for every  $H \in \mathcal{H}$ . Note that an  $H$ -free graph is also  $H$ -heavy, and if  $H_1$  is an induced subgraph of  $H_2$ , then an  $H_1$ -free ( $H_1$ -heavy) graph is also  $H_2$ -free ( $H_2$ -heavy).

The graph  $K_{1,3}$  is called the *claw*, its (only) vertex of degree 3 is called its *center*, and the other vertices are the *end vertices*. In this paper, instead of  $K_{1,3}$ -free ( $K_{1,3}$ -heavy), we use the terminology claw-free (claw-heavy).

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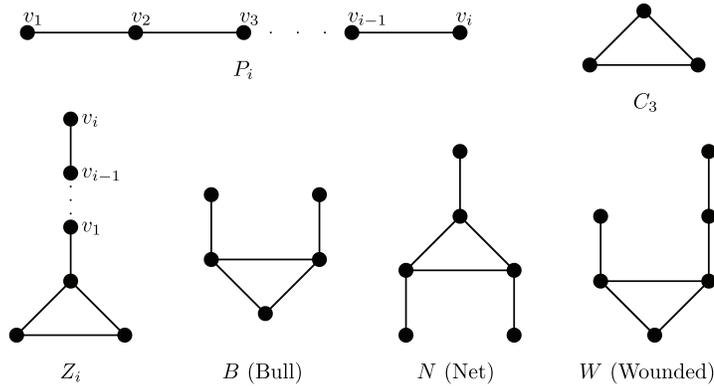


FIG. 1.1. Graphs  $P_i, C_3, Z_i, B, N$ , and  $W$ .

The following characterization of pairs of forbidden subgraphs for the existence of Hamilton cycles in graphs is well known.

**THEOREM 1.1** (see Bedrossian [1]). *Let  $R$  and  $S$  be connected graphs with  $R, S \neq P_3$ , and let  $G$  be a 2-connected graph. Then  $G$  being  $\{R, S\}$ -free implies that  $G$  is Hamiltonian if and only if (up to symmetry)  $R = K_{1,3}$  and  $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ , or  $W$  (see Figure 1.1).*

Our aim in this paper is to consider the corresponding heavy subgraph conditions for a graph to be Hamiltonian. First, we notice that every 2-connected  $P_3$ -heavy graph contains a Hamilton cycle. This can be easily deduced from the following result.

**THEOREM 1.2** (see Fan [5]). *Let  $G$  be a 2-connected graph. If  $\max\{d(u), d(v)\} \geq n/2$  for every pair of vertices at distance 2 in  $G$ , then  $G$  is Hamiltonian.*

It is not difficult to see that  $P_3$  is the only connected graph  $S$  such that every 2-connected  $S$ -heavy graph is Hamiltonian. So we have the following problem.

**PROBLEM 1.1.** *Which two connected graphs  $R$  and  $S$  other than  $P_3$  imply that every 2-connected  $\{R, S\}$ -heavy graph is Hamiltonian?*

By Theorem 1.1, we get that (up to symmetry)  $R = K_{1,3}$ , and  $S$  must be one of the graphs  $P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ , or  $W$ .

In this paper we prove the following results.

**THEOREM 1.3.** *If  $G$  is a 2-connected  $\{K_{1,3}, W\}$ -heavy graph, then  $G$  is Hamiltonian.*

**THEOREM 1.4.** *If  $G$  is a 2-connected  $\{K_{1,3}, N\}$ -heavy graph, then  $G$  is Hamiltonian.*

At the same time, we find a 2-connected  $\{K_{1,3}, P_6\}$ -heavy graph which is not Hamiltonian (see Figure 1.2).

We can also construct a 2-connected, claw-free, and  $P_6$ -heavy graph which is not Hamiltonian. This can be shown as follows: Let  $G$  be the graph in Figure 1.2, where  $r \geq 15$  is an integer divisible by 3. Let  $V_1, V_2, V_3$  be a balanced partition of  $K_r$ , and  $G'$  be the graph obtained from  $G$  by deleting all the edges in  $\bigcup_{i=1}^3 \{x_i v : v \in V_i\}$ . Then  $G'$  is a 2-connected, claw-free, and  $P_6$ -heavy graph which is not Hamiltonian.

Note that  $W$  contains induced copies of  $P_4, P_5, C_3, Z_1, Z_2$ , and  $B$ . So we have the following result.

**THEOREM 1.5.** *Let  $R$  and  $S$  be connected graphs with  $R, S \neq P_3$ , and let  $G$  be a 2-connected graph. Then  $G$  being  $\{R, S\}$ -heavy implies that  $G$  is Hamiltonian if and only if (up to symmetry)  $R = K_{1,3}$  and  $S = P_4, P_5, C_3, Z_1, Z_2, B, N$ , or  $W$ .*

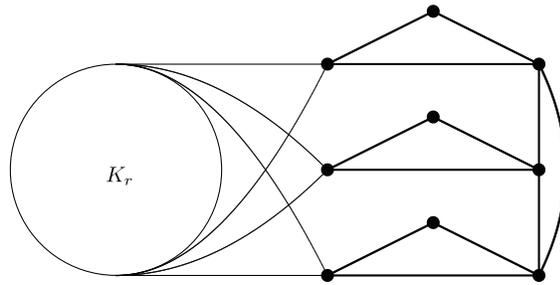


FIG. 1.2. A 2-connected  $\{K_{1,3}, P_6\}$ -heavy non-Hamiltonian graph ( $r \geq 5$ ).

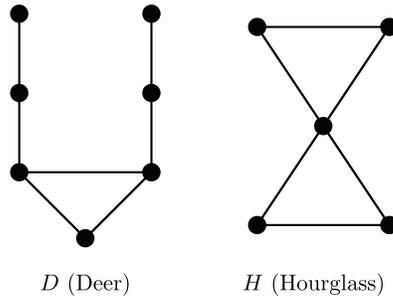


FIG. 1.3. Graphs  $D$  and  $H$ .

Thus, Theorem 1.5 gives a complete answer to Problem 1.1.

For claw-heavy graphs, Chen, Zhang, and Qiao get the following result.

**THEOREM 1.6** (see Chen, Zhang, and Qiao [4]). *Let  $G$  be a 2-connected graph. If  $G$  is claw-heavy and, moreover,  $\{P_7, D\}$ -free or  $\{P_7, H\}$ -free, then  $G$  is Hamiltonian (see Figure 1.3).*

It is clear that every  $P_6$ -free graph is also  $\{P_7, D\}$ -free. Thus we have that every 2-connected claw-heavy and  $P_6$ -free graph is Hamiltonian. Together with Theorems 1.3 and 1.4, we have the following characterization.

**THEOREM 1.7.** *Let  $S$  be a connected graph with  $S \neq P_3$ , and let  $G$  be a 2-connected claw-heavy graph. Then  $G$  being  $S$ -free implies that  $G$  is Hamiltonian if and only if  $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ , or  $W$ .*

The necessity of this theorem follows from Theorem 1.1 immediately.

It is known that the only 2-connected  $\{K_{1,3}, Z_3\}$ -free non-Hamiltonian graphs have nine vertices (see [6]); hence for  $n \geq 10$  every 2-connected  $\{K_{1,3}, Z_3\}$ -free graph of order  $n$  is also Hamiltonian. But this is not true for  $\{K_{1,3}, Z_3\}$ -heavy graphs. A counterexample is shown in Figure 1.4.

Instead of Theorems 1.3 and 1.4, we prove the following two stronger results.

**THEOREM 1.8.** *If  $G$  is a 2-connected  $\{K_{1,3}, N_{1,1,2}, D\}$ -heavy graph, then  $G$  is Hamiltonian (see Figure 1.5).*

**THEOREM 1.9.** *If  $G$  is a 2-connected  $\{K_{1,3}, N_{1,1,2}, H_{1,1}\}$ -heavy graph, then  $G$  is Hamiltonian (see Figure 1.5).*

Since a  $W$ -heavy graph is also  $\{N_{1,1,2}, D\}$ -heavy, Theorem 1.3 can be deduced from Theorem 1.8. Similarly, since an  $N$ -heavy graph is also  $\{N_{1,1,2}, H_{1,1}\}$ -heavy, Theorem 1.4 can be deduced from Theorem 1.9.

Note that Brousek [3] gave a complete characterization of triples of connected graphs  $K_{1,3}, X, Y$  such that a graph  $G$  being 2-connected and  $\{K_{1,3}, X, Y\}$ -free

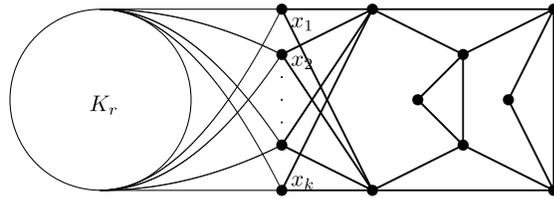


FIG. 1.4. A 2-connected  $\{K_{1,3}, Z_3\}$ -heavy non-Hamiltonian graph ( $k \geq 7, r \geq k + 4$ ).

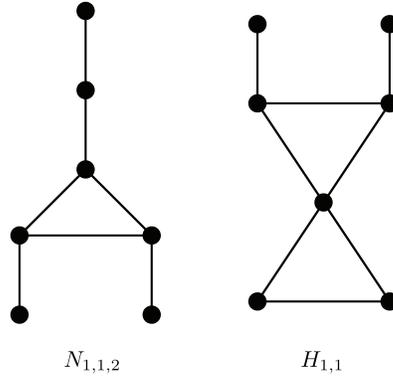


FIG. 1.5. Graphs  $N_{1,1,2}$  and  $H_{1,1}$ .

implies that  $G$  is Hamiltonian. Clearly, if  $K_{1,3}, S, T$  is a triple such that every 2-connected  $\{K_{1,3}, S, T\}$ -heavy graph is Hamiltonian, then, for some triple  $K_{1,3}, X, Y$  of [3],  $S$  and  $T$  are induced subgraphs of  $X$  and  $Y$ , respectively. (Of course, the triples of Theorems 1.8 and 1.9 have this property.) We refer an interested reader to [3] for more details.

**2. Some preliminaries.** We first give some additional terminology and notation.

Let  $G$  be a graph and  $X$  be a subset of  $V(G)$ . The subgraph of  $G$  induced by the set  $X$  is denoted  $G[X]$ . We use  $G - X$  to denote the subgraph induced by  $V(G) \setminus X$ .

Throughout this paper,  $k$  and  $\ell$  will always denote positive integers, and we use  $s$  and  $t$  to denote integers which may be nonpositive. For  $s \leq t$ , we use  $[x_s, x_t]$  to denote the set  $\{x_s, x_{s+1}, \dots, x_t\}$ . If  $[x_s, x_t]$  is a subset of the vertex set of a graph  $G$ , we use  $G[x_s, x_t]$ , instead of  $G[[x_s, x_t]]$ , to denote the subgraph induced by  $[x_s, x_t]$  in  $G$ .

For a path  $P$  and  $x, y \in V(P)$ ,  $P[x, y]$  denotes the subpath of  $P$  from  $x$  to  $y$ . Similarly, for a cycle  $C$  with a given orientation and  $x, y \in V(C)$ ,  $\vec{C}[x, y]$  or  $\overleftarrow{C}[y, x]$  denotes the  $(x, y)$ -path on  $C$  traversed in the same or opposite direction with respect to the given orientation of  $C$ .

Let  $G$  be a graph and  $x_1, x_2, y_1, y_2 \in V(G)$  with  $x_1 \neq x_2$  and  $y_1 \neq y_2$ . We define an  $(\{x_1, x_2\}, \{y_1, y_2\})$ -disjoint path pair, or briefly an  $(x_1x_2, y_1y_2)$ -pair, as a union of two vertex-disjoint paths  $P$  and  $Q$  such that

- (1) the origins of  $P$  and  $Q$  are in  $\{x_1, x_2\}$ , and
- (2) the termini of  $P$  and  $Q$  are in  $\{y_1, y_2\}$ .

If  $G$  is a graph on  $n \geq 2$  vertices,  $x \in V(G)$ , and a graph  $G'$  is obtained from  $G$  by adding a (new) vertex  $y$  and a pair of edges  $yx, yz$ , where  $z$  is an arbitrary vertex of  $G$ ,  $z \neq x$ , we say that  $G'$  is a 1-extension of  $G$  at  $x$  to  $y$ . Similarly, if  $x_1, x_2 \in V(G)$ ,  $x_1 \neq x_2$ , then the graph  $G'$  obtained from  $G$  by adding two (new) vertices  $y_1, y_2$  and the edges  $y_1x_1, y_2x_2$ , and  $y_1y_2$  is called the 2-extension of  $G$  at  $(x_1, x_2)$  to  $(y_1, y_2)$ .

Let  $G$  be a graph, and let  $u, v, w \in V(G)$  be distinct vertices of  $G$ . We say that  $G$  is  $(u, v, w)$ -composed (or briefly *composed*) if  $G$  has a spanning subgraph  $D$  (called the *carrier* of  $G$ ) such that there is an ordering  $v_{-k}, \dots, v_0, \dots, v_\ell$  ( $k, \ell \geq 1$ ) of  $V(D)$  ( $=V(G)$ ) and a sequence of graphs  $D_1, \dots, D_r$  ( $r \geq 1$ ) such that

- (1)  $u = v_{-k}, v = v_0, w = v_\ell$ ,
- (2)  $D_1$  is a triangle with  $V(D_1) = \{v_{-1}, v_0, v_1\}$ ,
- (3)  $V(D_i) = [v_{-k_i}, v_{\ell_i}]$  for some  $k_i, \ell_i, 1 \leq k_i \leq k, 1 \leq \ell_i \leq \ell$ , and  $D_{i+1}, 1 \leq i \leq r - 1$ , satisfies one of the following:
  - (a)  $D_{i+1}$  is a 1-extension of  $D_i$  at  $v_{-k_i}$  to  $v_{-k_i-1}$  or at  $v_{\ell_i}$  to  $v_{\ell_i+1}$ ,
  - (b)  $D_{i+1}$  is a 2-extension of  $D_i$  at  $(v_{-k_i}, v_{\ell_i})$  to  $(v_{-k_i-1}, v_{\ell_i+1})$ ,
- (4)  $D_r = D$ .

The ordering  $v_{-k}, \dots, v_0, \dots, v_\ell$  will be called a *canonical ordering* and the sequence  $D_1, \dots, D_r$  a *canonical sequence* of  $D$  (and also of  $G$ ). Note that a composed graph  $G$  can have several carriers, canonical orderings, and canonical sequences. Clearly, a composed graph  $G$  and its carrier  $D$  are 2-connected; for any canonical ordering,  $P = v_{-k} \cdots v_0 \cdots v_\ell$  is a Hamilton path in  $D$  (called a *canonical path*); and if  $D_1, \dots, D_r$  is a canonical sequence, then any  $D_i$  is  $(v_{-k_i}, v_0, v_{\ell_i})$ -composed,  $i = 1, \dots, r$ . Note that a  $(u, v, w)$ -composed graph is also  $(w, v, u)$ -composed.

Now we give a lemma on composed graphs which will be needed in our proofs.

LEMMA 2.1. *Let  $G$  be a composed graph, and let  $D$  and  $v_{-k}, \dots, v_0, \dots, v_\ell$  be a carrier and a canonical ordering of  $G$ . Then*

- (1)  $D$  has a Hamilton  $(v_0, v_{-k})$ -path,
- (2) for every  $v_s \in V(G) \setminus \{v_{-k}\}$ ,  $D$  has a spanning  $(v_0 v_\ell, v_s v_{-k})$ -pair.

*Proof.* Let  $D_1, \dots, D_r$  be a canonical sequence and  $Q$  the canonical path of  $D$  corresponding to the given ordering and, for every  $s \in [-k, \ell] \setminus \{0\}$ , let  $\hat{s}, 1 \leq \hat{s} \leq r$ , be the smallest integer for which  $v_s \in V(D_{\hat{s}})$ . Clearly,  $d_{D_{\hat{s}}}(v_s) = 2$ .

Now we prove (1) by induction on  $|V(D)|$ . If  $|V(D)| = 3$ , the assertion is trivially true. Suppose now that  $|V(D)| \geq 4$  and that the assertion is true for every graph with at most  $|V(D)| - 1$  vertices. By the definition of a carrier, we have the following two cases.

*Case 1.*  $V(D_{r-1}) = [v_{-k+1}, v_\ell]$  and  $D$  is a 1-extension of  $D_{r-1}$  at  $v_{-k+1}$  to  $v_{-k}$ .

By the induction hypothesis,  $D_{r-1}$  has a Hamilton  $(v_0, v_{-k+1})$ -path  $P'$ . Then  $P = v_0 P' v_{-k+1} v_{-k}$  is a Hamilton  $(v_0, v_{-k})$ -path in  $D$ .

*Case 2.*  $V(D_{r-1}) = [v_{-k}, v_{\ell-1}]$  and  $D$  is a 1-extension of  $D_{r-1}$  at  $v_{\ell-1}$  to  $v_\ell$ , or  $V(D_{r-1}) = [v_{-k+1}, v_{\ell-1}]$  and  $D$  is a 2-extension of  $D_{r-1}$  at  $(v_{-k+1}, v_{\ell-1})$  to  $(v_{-k}, v_\ell)$ . In this case,  $v_\ell$  has a neighbor  $v_s$  other than  $v_{\ell-1}$ , where  $s \in [-k, \ell - 2]$ .

*Case 2.1.*  $s \in [-k, -2]$ . In this case  $s + 1 \in [-k + 1, -1]$ . Consider the graph  $D' = D_{s+1}$ . Let  $V(D') = [v_{s+1}, v_t]$ , where  $t > 0$ . By the induction hypothesis, there exists a Hamilton  $(v_0, v_t)$ -path  $P'$  of  $D'$ . Then the path  $P = P' Q[v_t, v_\ell] v_\ell v_s Q[v_s, v_{-k}]$  is a Hamilton  $(v_0, v_{-k})$ -path of  $D$ .

*Case 2.2.*  $s = -1$ . In this case, the path  $P = Q[v_0, v_\ell] v_\ell v_{-1} Q[v_{-1}, v_{-k}]$  is a Hamilton  $(v_0, v_{-k})$ -path of  $D$ .

*Case 2.3.*  $s \in [0, \ell - 2]$ . In this case  $s + 1 \in [1, \ell - 1]$ . Consider the graph  $D' = D_{s+1}$ . Let  $V(D') = [v_t, v_{s+1}]$ , where  $t < 0$  and  $d_{D'}(v_{s+1}) = 2$ . By the induction hypothesis, there exists a Hamilton  $(v_0, v_t)$ -path  $P'$  of  $D'$ , and the edge  $v_s v_{s+1}$  is in  $E(P')$  by the fact  $d_{D'}(v_{s+1}) = 2$ . Thus the path  $P = P' - v_s v_{s+1} \cup Q[v_{s+1}, v_\ell] v_\ell v_s \cup Q[v_t, v_{-k}]$  is a Hamilton  $(v_0, v_{-k})$ -path of  $G$ .

So the proof of (1) is complete. Now we prove (2). We distinguish the following three cases.

*Case 1.*  $s \in [-k + 1, 0]$ . In this case,  $s - 1 \in [-k, -1]$ . Consider the graph  $D' = D_{s-1}$ . Let  $V(D') = [v_{s-1}, v_t]$ , where  $t > 0$  and  $d_{D'}(v_{s-1}) = 2$ . By (1), there exists a Hamilton  $(v_0, v_t)$ -path  $P'$  of  $D'$  and  $v_{s-1}v_s \in E(P')$ . Thus  $R' = P' - v_{s-1}v_s$  is a spanning  $(v_0v_t, v_s v_{s-1})$ -pair of  $D'$ , and  $R = R' \cup Q[v_t, v_l] \cup Q[v_{s-1}, v_{-k}]$  is a spanning  $(v_0v_\ell, v_s v_{-k})$ -pair of  $D$ .

*Case 2.*  $s = 1$ . In this case,  $R = Q[v_0, v_{-k}] \cup Q[v_1, v_\ell]$  is a spanning  $(v_0v_\ell, v_1v_{-k})$ -pair of  $D$ .

*Case 3.*  $s \in [2, \ell]$ . In this case,  $s - 1 \in [1, l - 1]$ . Consider the graph  $D' = D_{s-1}$ . Let  $V(D') = [v_t, v_{s-1}]$ , where  $t < 0$ . By (1), there exists a Hamilton  $(v_0, v_t)$ -path  $P'$  of  $G'$ . Thus  $P_1 = P'Q[v_t, v_{-k}]$  and  $P_2 = Q[v_s, v_\ell]$  form a spanning  $(v_0v_\ell, v_s v_{-k})$ -pair of  $D$ .

The proof is complete.  $\square$

Let  $G$  be a graph on  $n$  vertices and  $k \geq 3$  an integer. A sequence of vertices  $C = v_1v_2 \cdots v_kv_1$  such that for all  $i \in [1, k]$  either  $v_iv_{i+1} \in E(G)$  or  $d(v_i) + d(v_{i+1}) \geq n$  (indices are taken modulo  $k$ ) is called an *Ore-cycle* or briefly, *o-cycle* of  $G$ . The *deficit* of an *o-cycle*  $C$  is the integer  $\text{def}(C) = |\{i \in [1, k] : v_iv_{i+1} \notin E(G)\}|$ . Thus, a cycle is an *o-cycle* of deficit 0. We define an *o-path* of  $G$  similarly.

Now, we prove the following lemma on *o-cycles*.

LEMMA 2.2. *Let  $G$  be a graph, and let  $C'$  be an *o-cycle* in  $G$ . Then there is a cycle  $C$  in  $G$  such that  $V(C') \subset V(C)$ .*

*Proof.* Let  $C_1$  be an *o-cycle* in  $G$  such that  $V(C') \subset V(C_1)$  and  $\text{def}(C_1)$  is smallest possible, and suppose, to the contrary, that  $\text{def}(C_1) \geq 1$ . Without loss of generality suppose that  $C_1 = v_1v_2 \cdots v_kv_1$ , where  $v_1v_k \notin E(G)$  and  $d(v_1) + d(v_k) \geq n$ . We use  $P$  to denote the *o-path*  $P = v_1v_2 \cdots v_k$ .

If  $v_1$  and  $v_k$  have a common neighbor  $x \in V(G) \setminus V(P)$ , then  $C_2 = v_1Pv_kv_1$  is an *o-cycle* in  $G$  with  $V(C') \subset V(C_2)$  and  $\text{def}(C_2) < \text{def}(C_1)$ , a contradiction. Hence  $N_{G-P}(v_1) \cap N_{G-P}(v_k) = \emptyset$ . Then we have  $d_P(v_1) + d_P(v_k) \geq |V(P)|$  since  $d(v_1) + d(v_k) \geq n$ . Thus, there exists  $i \in [2, k - 1]$  such that  $v_i \in N_P(v_1)$  and  $v_{i-1} \in N_P(v_k)$ , and then again  $C_2 = v_1P[v_1, v_{i-1}]v_{i-1}v_kP[v_k, v_i]v_iv_1$  is an *o-cycle* with  $V(C') \subset V(C_2)$  and  $\text{def}(C_2) < \text{def}(C_1)$ , a contradiction.  $\square$

Note that Lemma 2.2 immediately implies that if  $P$  is an  $(x, y)$ -path or an *o-path* in  $G$  with  $|V(P)|$  larger than the length of a longest cycle in  $G$ , then  $xy \notin E(G)$  and  $d(x) + d(y) < n$ .

In the following, we denote  $\tilde{E}(G) = \{uv : uv \in E(G) \text{ or } d(u) + d(v) \geq n\}$ .

Let  $C$  be a cycle in  $G$ ;  $x, x_1, x_2 \in V(C)$  be three distinct vertices; and set  $X = V(Q)$ , where  $Q$  is the  $(x_1, x_2)$ -path on  $C$  containing  $x$ . We say that the pair of vertices  $(x_1, x_2)$  is *x-good on  $C$*  if for some  $j \in \{1, 2\}$  there is a vertex  $x' \in X \setminus \{x_j\}$  such that

- (1) there is an  $(x, x_{3-j})$ -path  $P$  such that  $V(P) = X \setminus \{x_j\}$ ,
- (2) there is an  $(xx_{3-j}, x'x_j)$ -pair  $D$  such that  $V(D) = X$ ,
- (3)  $d(x_j) + d(x') \geq n$ .

LEMMA 2.3. *Let  $G$  be a graph and  $C$  be a cycle of  $G$  with a given orientation. Let  $x, y \in V(C)$ , and let  $R$  be an  $(x, y)$ -path in  $G$  which is internally disjoint from  $C$ . If there are vertices  $x_1, x_2, y_1, y_2 \in V(C) \setminus \{x, y\}$  such that*

- (1)  $x_2, x, x_1, y_1, y, y_2$  appear in this order along  $\vec{C}$  (possibly  $x_1 = y_1$  or  $x_2 = y_2$ ),
- (2)  $(x_1, x_2)$  is *x-good on  $C$* ,
- (3)  $(y_1, y_2)$  is *y-good on  $C$* ,

*then there is a cycle  $C'$  in  $G$  such that  $V(C) \cup V(R) \subset V(C')$ .*

*Proof.* Assume the opposite. Let  $P_1$  and  $D_1$  be the path and disjoint path pair associated with  $x$ , and  $P_2$  and  $D_2$  those associated with  $y$ ; and let  $Q_1 = \overrightarrow{C}[x_1, y_1]$  and  $Q_2 = \overleftarrow{C}[x_2, y_2]$ .

By the definition of an  $x$ -good pair, without loss of generality, we can assume that  $P_1$  is an  $(x, x_1)$ -path,  $D_1$  is an  $(xx_1, x'x_2)$ -pair, and  $d(x_2) + d(x') \geq n$ .

*Case 1.*  $P_2$  is a  $(y, y_1)$ -path,  $D_2$  is a  $(yy_1, y'y_2)$ -pair, and  $d(y_2) + d(y') \geq n$ .

In this case the path  $P = Q_2 \cup D_2 \cup R \cup P_1 \cup Q_1$  is an  $(x_2, y')$ -path which contains all the vertices in  $V(C) \cup V(R)$ , and  $P' = Q_2 \cup D_1 \cup R \cup P_2 \cup Q_1$  is an  $(x', y_2)$ -path which contains all the vertices in  $V(C) \cup V(R)$ . Thus, by Lemma 2.2,  $d(x_2) + d(y') < n$  and  $d(x') + d(y_2) < n$ , a contradiction to  $d(x_2) + d(x') \geq n$  and  $d(y_2) + d(y') \geq n$ .

*Case 2.*  $P_2$  is a  $(y, y_2)$ -path,  $D_2$  is a  $(yy_2, y'y_1)$ -pair, and  $d(y_1) + d(y') \geq n$ .

*Case 2.1.* The  $(xx_1, x'x_2)$ -pair  $D_1$  is formed by an  $(x, x_2)$ -path and an  $(x_1, x')$ -path.

In this case, the path  $P = Q_2 \cup P_2 \cup R \cup P_1 \cup Q_1$  is an  $(x_2, y_1)$ -path which contains all the vertices in  $V(C) \cup V(R)$ , and the path  $P' = D_1 \cup Q_1 \cup Q_2 \cup R \cup D_2$  is an  $(x', y')$ -path which contains all the vertices in  $V(C) \cup V(R)$ . By Lemma 2.2,  $d(x_2) + d(y_1) < n$  and  $d(x') + d(y') < n$ , a contradiction.

*Case 2.2.* The  $(xx_1, x'x_2)$ -pair  $D_1$  is formed by an  $(x, x')$ -path and an  $(x_1, x_2)$ -path.

*Case 2.2.1.* The  $(yy_2, y'y_1)$ -pair  $D_2$  is formed by an  $(y, y_1)$ -path and an  $(y_2, y')$ -path.

This case can be proved similarly as in Case 2.1.

*Case 2.2.2.* The  $(yy_2, y'y_1)$ -pair  $D_2$  is formed by an  $(y, y')$ -path and an  $(y_1, y_2)$ -path.

In this case, the path  $P = Q_2 \cup D_2 \cup R \cup P_1 \cup Q_1$  is an  $(x_2, y')$ -path containing all vertices in  $V(C) \cup V(R)$ , and the path  $P' = Q_2 \cup D_1 \cup R \cup P_2 \cup Q_1$  is an  $(x', y_1)$ -path containing all vertices in  $V(C) \cup V(R)$ . By Lemma 2.2,  $d(x_2) + d(y') < n$  and  $d(x') + d(y_1) < n$ , a contradiction.

The proof is complete.  $\square$

**3. Proof of Theorem 1.8.** Let  $C$  be a longest cycle of  $G$ . Set  $n = |V(G)|$  and  $c = |V(C)|$ , and assume that  $G$  is not Hamiltonian, i.e.,  $c < n$ . Then  $V(G) \setminus V(C) \neq \emptyset$ . Since  $G$  is 2-connected, there exists a  $(u_0, v_0)$ -path with length at least 2 which is internally disjoint from  $C$ , where  $u_0, v_0 \in V(C)$ . Let  $R = z_0 z_1 z_2 \cdots z_{r+1}$ , where  $z_0 = u_0$  and  $z_{r+1} = v_0$ , be such a path, and choose  $R$  as short as possible. Let  $r_1$  and  $r_2$  denote the number of interior vertices in the two subpaths of  $C$  from  $u_0$  to  $v_0$  (note that clearly  $r_1 + r_2 + 2 = c$ ). We specify an orientation of  $C$ , and label the vertices of  $C$  using two distinct notations  $u_i$  and  $v_i$ ,  $-r_2 \leq i \leq r_1$ , such that  $\overrightarrow{C} = u_0 u_1 u_2 \cdots u_{r_1} v_0 u_{-r_2} u_{-r_2+1} \cdots u_{-1} u_0$  and  $\overleftarrow{C} = v_0 v_1 v_2 \cdots v_{r_1} u_0 v_{-r_2} v_{-r_2+1} \cdots v_{-1} v_0$ , where  $u_\ell = v_{r_1+1-\ell}$  and  $u_{-k} = v_{-r_2-1+k}$  (see Figure 3.1). Let  $H$  be the component of  $G - C$  which contains the vertices in  $[z_1, z_r]$ .

CLAIM 1. *Let  $x \in V(H)$  and  $y \in \{u_1, u_{-1}, v_1, v_{-1}\}$ . Then  $xy \notin \tilde{E}(G)$ .*

*Proof.* Without loss of generality, we assume  $y = u_1$ . Let  $P'$  be an  $(x, z_1)$ -path in  $H$ . Then  $P = P' z_1 u_0 \overrightarrow{C}[u_0, u_1]$  is an  $(x, y)$ -path which contains all the vertices in  $V(C) \cup V(P')$ . By Lemma 2.2, we have  $xy \notin \tilde{E}(G)$ .  $\square$

CLAIM 2.  $u_1 u_{-1} \in \tilde{E}(G)$ ,  $v_1 v_{-1} \in \tilde{E}(G)$ .

*Proof.* If  $u_1 u_{-1} \notin E(G)$ , by Claim 1, the graph induced by  $\{u_0, z_1, u_1, u_{-1}\}$  is a claw, where  $d(z_1) + d(u_{\pm 1}) < n$ . Since  $G$  is a claw-heavy graph, we have that  $d(u_1) + d(u_{-1}) \geq n$ .

The second assertion can be proved similarly.  $\square$

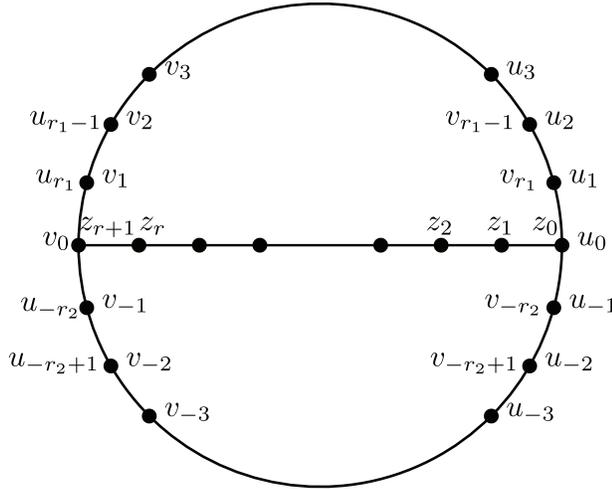


FIG. 3.1.  $C \cup R$ , the subgraph of  $G$ .

CLAIM 3.  $u_1v_{-1} \notin \tilde{E}(G)$ ,  $u_{-1}v_1 \notin \tilde{E}(G)$ ,  $u_0v_{\pm 1} \notin \tilde{E}(G)$ ,  $u_{\pm 1}v_0 \notin \tilde{E}(G)$ .

*Proof.* Since  $P = \overrightarrow{C}[u_1, v_0]R\overleftarrow{C}[u_0, v_{-1}]$  is a  $(u_1, v_{-1})$ -path which contains all the vertices in  $V(C) \cup V(R)$ , we have  $u_1v_{-1} \notin \tilde{E}(G)$  by Lemma 2.2.

If  $u_0v_1 \in \tilde{E}(G)$ , then  $C' = \overrightarrow{C}[u_1, v_1]v_1u_0R\overrightarrow{C}[v_0, u_{-1}]u_{-1}u_1$  is an  $o$ -cycle which contains all the vertices of  $V(C) \cup V(R)$ . By Lemma 2.2, there exists a cycle which contains all the vertices in  $V(C) \cup V(R)$ , a contradiction.

The other assertions can be proved similarly.  $\square$

CLAIM 4. Either  $u_1u_{-1} \in E(G)$  or  $v_1v_{-1} \in E(G)$ .

*Proof.* Assume the opposite. By Claim 2 we have  $d(u_1) + d(u_{-1}) \geq n$  and  $d(v_1) + d(v_{-1}) \geq n$ . By Claim 3, we have  $d(u_1) + d(v_{-1}) < n$  and  $d(u_{-1}) + d(v_1) < n$ , a contradiction.  $\square$

Now, we distinguish two cases.

Case 1.  $r \geq 2$ , or  $r = 1$  and  $u_0v_0 \notin E(G)$ .

By Claim 4, without loss of generality, we assume that  $u_1u_{-1} \in E(G)$ . Thus  $G[u_{-1}, u_1]$  is  $(u_{-1}, u_0, u_1)$ -composed.

CLAIM 5.  $z_2u_0 \notin \tilde{E}(G)$ .

*Proof.* By the choice of the path  $R$ , we have  $z_2u_0 \notin E(G)$ . Now we prove that  $d(z_2) + d(u_0) < n$ .

CLAIM 5.1. Every neighbor of  $u_0$  is in  $V(C) \cup V(H)$ ; every neighbor of  $z_2$  is in  $V(C) \cup V(H)$ .

*Proof.* Assume the opposite. Let  $z' \in V(H')$  be a neighbor of  $u_0$ , where  $H'$  is a component of  $G - C$  other than  $H$ . Then we have  $z'u_0 \notin E(G)$  and  $N_{G-C}(z') \cap N_{G-C}(z_1) = \emptyset$ .

By Claim 1, we have  $u_1z_1 \notin \tilde{E}(G)$ , and similarly  $u_1z' \notin \tilde{E}(G)$ . Thus the graph induced by  $\{u_0, u_1, z_1, z'\}$  is a claw, where  $d(u_1) + d(z_1) < n$  and  $d(u_1) + d(z') < n$ . Then we have  $d(z_1) + d(z') \geq n$ .

Since  $N_{G-C}(z_1) \cap N_{G-C}(z') = \emptyset$ , there exist two vertices  $x_1, x_2 \in V(C)$  such that  $x_1x_2 \in E(\overrightarrow{C})$  and  $z_1x_1, z'x_2 \in E(G)$ . Thus  $P = z_1x_1\overrightarrow{C}[x_1, x_2]x_2z'$  is a  $(z_1, z')$ -path which contains all the vertices in  $V(C) \cup \{z_1, z'\}$ . By Lemma 2.2, there exists a cycle which contains all the vertices in  $V(C) \cup \{z_1, z'\}$ , a contradiction.

If  $z_2 = v_0$ , the second assertion can be proved similarly, and if  $z_2 \neq v_0$ , the assertion is obvious.  $\square$

Let  $h = |V(H)|$  and  $k = |N_H(u_0)|$ . Then we have  $d_H(z_2) + d_H(u_0) \leq h + k$ . Since  $z_1 \in N_H(u_0)$ , we have  $k \geq 1$ . Let  $N_H(u_0) = \{y_1, y_2, \dots, y_k\}$ , where  $y_1 = z_1$ .

CLAIM 5.2.  $y_i y_j \in \widetilde{E}(G)$  for all  $1 \leq i < j \leq k$ .

*Proof.* If  $y_i y_j \notin E(G)$ , then by Claim 1, the graph induced by  $\{u_0, u_1, y_i, y_j\}$  is a claw, where  $d(y_i) + d(u_1) < n$  and  $d(y_j) + d(u_1) < n$ . Thus we have  $d(y_i) + d(y_j) \geq n$ .  $\square$

Now, let  $Q$  be the  $o$ -path  $Q = z_2 y_1 y_2 \cdots y_k u_0$ . It is clear that  $R[z_2, v_0]$  and  $Q$  are internally disjoint, and  $Q$  contains at least  $k$  vertices in  $V(H)$ . In the following, we use  $C'$  to denote the cycle  $\vec{C}[u_1, u_{-1}]u_{-1}u_1$  if  $z_2 \neq v_0$ , and to denote the  $o$ -cycle  $\vec{C}[u_1, v_1]v_1v_{-1}\vec{C}[v_{-1}, u_{-1}]u_{-1}u_1$  if  $z_2 = v_0$ .

By Claims 1 and 3, we have  $z_2 v_{r_1} \notin E(G)$ , where  $v_{r_1} = u_1$ . Let  $v_\ell$  be the last vertex in  $\vec{C}[v_1, u_1]$  such that  $z_2 v_\ell \in E(G)$ . If there are no neighbors of  $z_2$  in  $\vec{C}[v_1, u_1]$ , then let  $v_\ell = v_0$ .

CLAIM 5.3. For every vertex  $v_{\ell'} \in N_{[v_1, v_{r_1}]}(z_2) \cup \{v_0\}$ ,  $u_0 v_{\ell'+1} \notin E(G)$ .

*Proof.* By Claim 3, we have  $u_0 v_1 \notin E(G)$ .

If  $z_2 v_{\ell'} \in E(G)$  and  $u_0 v_{\ell'+1} \in E(G)$ , then  $C'' = \vec{C}[v_{\ell'}, v_{\ell'+1}]v_{\ell'+1}u_0 Q z_2 v_{\ell'}$  is an  $o$ -cycle which contains all the vertices in  $V(C) \cup V(Q)$ , a contradiction.  $\square$

CLAIM 5.4.  $r_1 - \ell \geq k + 1$ , and for every vertex  $v_{\ell'} \in [v_{\ell+1}, v_{\ell+k}]$ ,  $u_0 v_{\ell'} \notin E(G)$ .

*Proof.* Assume the opposite. Let  $v_{\ell'}$  be the first vertex in  $[v_{\ell+1}, v_{r_1}]$  such that  $u_0 v_{\ell'} \in E(G)$ , and let  $\ell' - \ell < k + 1$ .

If  $v_\ell = v_0$ , then  $C''' = \vec{C}[v_0, u_{-1}]u_{-1}u_1 \vec{C}[u_1, v_{\ell'}]v_{\ell'}u_0 Q R[z_2, v_0]$  is an  $o$ -cycle which contains all the vertices in  $V(C) \setminus [v_1, v_{\ell'-1}] \cup V(Q)$ , and  $|V(C''')| > c$ , a contradiction.

Thus, we assume that  $v_\ell \neq v_0$  and  $z_2 v_\ell \in E(G)$ . Then  $C''' = \vec{C}[v_\ell, v_{\ell'}]v_{\ell'}u_0 Q z_2 v_\ell$  is an  $o$ -cycle which contains all the vertices in  $V(C) \setminus [v_{\ell+1}, v_{\ell'-1}] \cup V(Q)$ , and  $|V(C''')| > c$ , a contradiction.

Thus we have  $\ell' - \ell \geq k + 1$ . Note that  $u_0 v_{r_1} \in E(G)$ , and we have  $r_1 - \ell \geq k + 1$ .  $\square$

Let  $d_1 = |N_{[v_1, v_{r_1}]}(z_2) \cup \{v_0\}|$ ,  $d_2 = |N_{[v_{-r_2}, v_{-1}]}(z_2) \cup \{v_0\}|$ ,  $d'_1 = |N_{[v_1, v_{r_1}]}(u_0)|$  and  $d'_2 = |N_{[v_{-r_2}, v_{-1}]}(u_0)|$ . Then  $d_C(z_2) \leq d_1 + d_2 - 1$  and  $d_C(u_0) \leq d'_1 + d'_2 + 1$ .

By Claims 5.3 and 5.4, we have  $d'_1 \leq r_1 - d_1 - k + 1$ , and similarly,  $d'_2 \leq r_2 - d_2 - k + 1$ . Thus  $d_C(z_2) + d_C(u_0) \leq r_1 + r_2 - 2k + 2 = c - 2k$ . Note that  $d_H(z_2) + d_H(u_0) \leq h + k$ . By Claim 5.1, we have  $d(z_2) + d(u_0) \leq c + h - k < n$ .  $\square$

Recall that  $G[u_{-1}, u_1]$  is  $(u_{-1}, u_0, u_1)$ -composed. Now we prove the following claims.

CLAIM 6. If  $G[u_{-k}, u_\ell]$  is  $(u_{-k}, u_0, u_\ell)$ -composed with canonical ordering  $u_{-k}, u_{-k+1}, \dots, u_\ell$ , then  $k \leq r_2 - 2$  and  $\ell \leq r_1 - 2$ .

*Proof.* Let  $D_1, D_2, \dots, D_r$  be a canonical sequence of  $G[u_{-k}, u_\ell]$  corresponding to the canonical ordering  $u_{-k}, u_{-k+1}, \dots, u_\ell$ . Suppose that  $k > r_2 - 2$ . Consider the graph  $D' = D_{-r_2+1}$ , where  $-r_2 + 1$  is the smallest integer such that  $u_{-r_2+1} \in V(D_{-r_2+1})$ . Let  $V(D') = [u_{-r_2+1}, u_{\ell'}]$ . By Lemma 2.1, there exists a  $(u_0, u_{\ell'})$ -path  $P$  such that  $V(P) = [u_{-r_2+1}, u_{\ell'}]$ . Then  $C' = v_{-1}v_0 R P \vec{C}[u_{\ell'}, v_1]v_1v_{-1}$  is an  $o$ -cycle which contains all the vertices in  $V(C) \cup V(R)$ , a contradiction.  $\square$

CLAIM 7. If  $G[u_{-k}, u_\ell]$  is  $(u_{-k}, u_0, u_\ell)$ -composed with canonical ordering  $u_{-k}, u_{-k+1}, \dots, u_\ell$ , where  $k \leq r_2 - 2$  and  $\ell \leq r_1 - 2$ , and any two nonadjacent vertices in  $[u_{-k-1}, u_{\ell+1}]$  have degree sum less than  $n$ , then one of the following is true:

- (1)  $G[u_{-k-1}, u_\ell]$  is  $(u_{-k-1}, u_0, u_\ell)$ -composed with canonical ordering  $u_{-k-1}, u_{-k}, \dots, u_\ell$ ,
- (2)  $G[u_{-k}, u_{\ell+1}]$  is  $(u_{-k}, u_0, u_{\ell+1})$ -composed with canonical ordering  $u_{-k}, u_{-k+1}, \dots, u_{\ell+1}$ ,
- (3)  $G[u_{-k-1}, u_{\ell+1}]$  is  $(u_{-k-1}, u_0, u_{\ell+1})$ -composed with canonical ordering  $u_{-k-1}, u_{-k}, \dots, u_{\ell+1}$ .

*Proof.* Assume the opposite, which implies that for every vertex  $u_s \in [u_{-k+1}, u_\ell]$ ,  $u_{-k-1}u_s \notin E(G)$ , and for every vertex  $u_s \in [u_{-k}, u_{\ell-1}]$ , we have  $u_{\ell+1}u_s \notin E(G)$  and  $u_{-k-1}u_{\ell+1} \notin E(G)$ .

CLAIM 7.1. *Let  $z \in \{z_1, z_2\}$  and  $u_s \in [u_{-k-1}, u_{\ell+1}] \setminus \{u_0\}$ . Then  $zu_s \notin \tilde{E}(G)$ .*

*Proof.* Without loss of generality, we assume that  $s > 0$ . If  $s = 1$ , the assertion is true by Claims 1 and 3. So we assume that  $s \in [2, \ell + 1]$  and  $s - 1 \in [1, \ell]$ . By the definition of a composed graph, there exists  $t \in [-k, -1]$  such that  $G[u_t, u_{s-1}]$  is  $(u_t, u_0, u_{s-1})$ -composed. By Lemma 2.1, there exists a  $(u_0, u_t)$ -path  $P'$  such that  $V(P') = [u_t, u_{s-1}]$ .

If  $z \neq v_0$ , then  $P = R[z, u_0]P'\overleftarrow{C}[u_t, u_s]$  is a  $(z, u_s)$ -path which contains all the vertices in  $V(C) \cup \{z\}$ . By Lemma 2.2, we have  $zu_s \notin \tilde{E}(G)$ .

If  $z = v_0$  and  $v_0u_s \in \tilde{E}(G)$ , then  $C' = RP'\overleftarrow{C}[u_t, v_{-1}]v_{-1}v_1\overleftarrow{C}[v_1, u_s]u_sv_0$  is an  $o$ -cycle which contains all the vertices in  $V(C) \cup V(R)$ , a contradiction.  $\square$

Let  $G' = G[[u_{-k-1}, u_\ell] \cup \{z_1, z_2\}]$  and  $G'' = G[[u_{-k-1}, u_{\ell+1}] \cup \{z_1, z_2\}]$ .

CLAIM 7.2.  *$G''$ , and then  $G'$ , is  $\{K_{1,3}, N_{1,1,2}\}$ -free.*

*Proof.* By Claims 5 and 7.1, and the condition that any two nonadjacent vertices in  $[u_{-k-1}, u_{\ell+1}]$  have degree sum less than  $n$ , we have that any two nonadjacent vertices in  $G''$  have degree sum less than  $n$ . Since  $G$  (and then  $G''$ ) is  $\{K_{1,3}, N_{1,1,2}\}$ -heavy, we have that  $G''$  is  $\{K_{1,3}, N_{1,1,2}\}$ -free.  $\square$

CLAIM 7.3.  *$N_{G'}(u_0) \setminus \{z_1\}$  is a clique.*

*Proof.* If there are two vertices  $x, x' \in N_{G'}(u_0) \setminus \{z_1\}$  such that  $xx' \notin E(G')$ , then the graph induced by  $\{u_0, z_1, x, x'\}$  is a claw, a contradiction.  $\square$

Now, we define  $N_i = \{x \in V(G') : d_{G'}(x, u_{-k-1}) = i\}$ . Then we have  $N_0 = \{u_{-k-1}\}$ ,  $N_1 = \{u_{-k}\}$ , and  $N_2 = N_{G'}(u_{-k}) \setminus \{u_{-k-1}\}$ .

By the definition of a composed graph, we have  $|N_2| \geq 2$ . If there are two vertices  $x, x' \in N_2$  such that  $xx' \notin E(G')$ , then the graph induced by  $\{u_{-k}, u_{-k-1}, x, x'\}$  is a claw, a contradiction. Thus,  $N_2$  is a clique.

We assume  $u_0 \in N_j$ , where  $j \geq 2$ . Then  $z_1 \in N_{j+1}$  and  $z_2 \in N_{j+2}$ .

If  $|N_i| = 1$  for some  $i \in [2, j - 1]$ , say,  $N_i = \{x\}$ , then  $x$  is a cut vertex of the graph  $G[u_{-k}, u_i]$ . By the definition of a composed graph,  $G[u_{-k}, u_i]$  is 2-connected. This implies  $|N_i| \geq 2$  for every  $i \in [2, j - 1]$ .

CLAIM 7.4. *For  $i \in [1, j]$ ,  $N_i$  is a clique.*

*Proof.* We prove this claim by induction on  $i$ . For  $i = 1, 2$ , the claim is true by the analysis above. So we assume that  $3 \leq i \leq j$ , and we have that  $N_{i-3}, N_{i-2}, N_{i-1}, N_{i+1}$ , and  $N_{i+2}$  is nonempty, and  $|N_{i-1}| \geq 2$ .

First we choose a vertex  $x \in N_i$  which has a neighbor  $y \in N_{i+1}$  such that it has a neighbor  $z \in N_{i+2}$ . We prove that for every  $x' \in N_i$ ,  $xx' \in E(G)$ . We assume that  $xx' \notin E(G)$ .

If  $x'y \in E(G)$ , then the graph induced by  $\{y, x, x', z\}$  is a claw, a contradiction. Thus, we have  $x'y \notin E(G)$ . If  $x$  and  $x'$  have a common neighbor in  $N_{i-1}$ , denote it by  $w$ ; then let  $v$  be a neighbor of  $w$  in  $N_{i-2}$ , and the graph induced by  $\{w, v, x, x'\}$  is a claw, a contradiction. Thus we have that  $x$  and  $x'$  have no common neighbors in  $N_{i-1}$ .

Let  $w$  be a neighbor of  $x$  in  $N_{i-1}$ , and  $w'$  be a neighbor of  $x'$  in  $N_{i-1}$ . Then  $xw', x'w \notin E(G)$ . Let  $v$  be a neighbor of  $w$  in  $N_{i-2}$ , and  $u$  be a neighbor of  $v$  in  $N_{i-3}$ . If  $w'v \notin E(G)$ , then the graph induced by  $\{w, v, w', x\}$  is a claw, a contradiction. Thus we have  $w'v \in E(G)$ , and then the graph induced by  $\{v, u, w', x', w, x, y\}$  is an  $N_{1,1,2}$ , a contradiction.

Thus we have  $xx' \in E(G)$  for every  $x' \in N_i$ .

Now, let  $x'$  and  $x''$  be two vertices in  $N_i$  other than  $x$  such that  $x'x'' \notin E(G)$ . We have  $xx', xx'' \in E(G)$ .

If  $x'y \in E(G)$ , then similarly to the case of  $x$ , we have  $x'x'' \in E(G)$ , a contradiction. Thus we have  $x'y \notin E(G)$ . Similarly,  $x''y \notin E(G)$ . Then the graph induced by  $\{x, x', x'', y\}$  is a claw, a contradiction.

Thus,  $N_i$  is a clique.  $\square$

If there exists some vertex  $y \in N_{j+1}$  other than  $z_1$ , then we have  $yu_0 \notin E(G)$  by Claim 7.3. Let  $x$  be a neighbor of  $y$  in  $N_j$ ,  $w$  be a neighbor of  $u_0$  in  $N_{j-1}$ , and  $v$  be a neighbor of  $w$  in  $N_{j-2}$ . Then  $xu_0 \in E(G)$  by Claim 7.4 and  $xw \in E(G)$  by Claim 7.3. Thus the graph induced by  $\{w, v, x, y, u_0, z_1, z_2\}$  is an  $N_{1,1,2}$ , a contradiction. So we assume that all vertices in  $[u_{-k}, u_\ell]$  are in  $\bigcup_{i=1}^j N_i$ .

If  $u_\ell \in N_j$ , then let  $w$  be a neighbor of  $u_0$  in  $N_{j-1}$ , and  $v$  be a neighbor of  $w$  in  $N_{j-2}$ . Then the graph induced by  $\{w, v, u_0, z_1, u_\ell, u_{\ell+1}\}$  is an  $N_{1,1,2}$ , a contradiction. Thus we have that  $u_\ell \notin N_j$  and then  $j \geq 3$ .

Let  $u_\ell \in N_i$ , where  $i \in [2, j - 1]$ . If  $u_\ell$  has a neighbor in  $N_{i+1}$ , then let  $y$  be a neighbor of  $u_\ell$  in  $N_{i+1}$ , and  $w$  be a neighbor of  $u_\ell$  in  $N_{i-1}$ . Then the graph induced by  $\{u_\ell, w, y, u_{\ell+1}\}$  is a claw, a contradiction. So we have that  $u_\ell$  has no neighbors in  $N_{i+1}$ .

Let  $x \in N_i$  be a vertex other than  $u_\ell$  which has a neighbor  $y$  in  $N_{i+1}$  such that it has a neighbor  $z$  in  $N_{i+2}$ . Let  $w$  be a neighbor of  $x$  in  $N_{i-1}$ , and  $v$  be a neighbor of  $w$  in  $N_{i-2}$ . If  $u_\ell w \notin E(G)$ , then the graph induced by  $\{x, w, u_\ell, y\}$  is a claw, a contradiction. So we have that  $u_\ell w \in E(G)$ . Then the graph induced by  $\{w, v, u_\ell, u_{\ell+1}, x, y, z\}$  is an  $N_{1,1,2}$ , a contradiction.

Thus the claim holds.  $\square$

Now we choose  $k, \ell$  such that

- (1)  $G[u_{-k}, u_\ell]$  is  $(u_{-k}, u_0, u_\ell)$ -composed with canonical ordering  $u_{-k}, u_{-k+1}, \dots, u_\ell$ ;
- (2) any two nonadjacent vertices in  $[u_{-k}, u_\ell]$  have degree sum less than  $n$ ; and
- (3)  $k + \ell$  is as big as possible.

By Claim 7, we have that there exists a vertex  $u_s \in [u_{-k+1}, u_\ell]$  such that  $d(u_{-k-1}) + d(u_s) \geq n$ , or there exists a vertex  $u_s \in [u_{-k}, u_{\ell-1}]$  such that  $d(u_s) + d(u_{\ell+1}) \geq n$ , or  $d(u_{-k-1}) + d(u_{\ell+1}) \geq n$ . Thus, we have the next result.

CLAIM 8.  $(u_{-k-1}, u_\ell)$  or  $(u_{-k}, u_{\ell+1})$  or  $(u_{-k-1}, u_{\ell+1})$  is  $u_0$ -good on  $C$ .

*Proof.* If there exists a vertex  $u_s \in [u_{-k+1}, u_\ell]$  such that  $d(u_{-k-1}) + d(u_s) \geq n$ , then, by Lemma 2.1, there exists a  $(u_0, u_\ell)$ -path  $P$  such that  $V(P) = [u_{-k}, u_\ell]$ , there exists a  $(u_0 u_\ell, u_s u_{-k})$ -pair  $D'$  such that  $V(D') = [u_{-k}, u_\ell]$ , and  $D = D' + u_{-k} u_{-k-1}$  is a  $(u_0 u_\ell, u_s u_{-k-1})$ -pair such that  $V(D) = [u_{-k-1}, u_\ell]$ . Thus  $(u_{-k-1}, u_\ell)$  is  $u_0$ -good on  $C$ .

If there exists a vertex  $u_s \in [u_{-k}, u_{\ell-1}]$  such that  $d(u_s) + d(u_{\ell+1}) \geq n$ , we can prove the result similarly.

If  $d(u_{-k-1}) + d(u_{\ell+1}) \geq n$ , then by Lemma 2.1, there exists a  $(u_0, u_\ell)$ -path  $P'$  such that  $V(P') = [u_{-k}, u_\ell]$ , and there exists a  $(u_0, u_{-k})$ -path  $P''$  such that  $V(P'') = [u_{-k}, u_\ell]$ . Then  $P = P' u_1 u_{\ell+1}$  is a  $(u_0, u_{\ell+1})$ -path such that  $V(P) = [u_{-k}, u_{\ell+1}]$ , and  $D = P'' u_{-k} u_{-k-1} \cup u_{\ell+1}$  is a  $(u_0 u_{\ell+1}, u_{\ell+1} u_{-k-1})$ -pair such that  $V(D) = [u_{-k-1}, u_{\ell+1}]$ . Thus  $(u_{-k-1}, u_{\ell+1})$  is  $u_0$ -good on  $C$ .  $\square$

CLAIM 9. *There exist  $v_{-k'} \in V(\overrightarrow{C}[v_{-1}, u_{-k-1}])$  and  $v_{\ell'} \in V(\overleftarrow{C}[v_1, u_{\ell+1}])$  such that  $(v_{-k'}, v_{\ell'})$  is  $v_0$ -good on  $C$ .*

*Proof.* By Claim 6, we have  $k \leq r_2 - 2$  and  $l \leq r_1 - 2$ .

If  $v_1 v_{-1} \notin E(G)$ , then by Claim 2,  $d(v_1) + d(v_{-1}) \geq n$ . Then  $P = v_0 v_1$  is a  $(v_0, v_1)$ -path and  $D = v_0 v_{-1} \cup v_1$  is a  $(v_0 v_1, v_{-1} v_1)$ -pair. Thus we have that  $(v_{-1}, v_1)$  is  $v_0$ -good on  $C$ .

Now we assume that  $v_1 v_{-1} \in E(G)$ , and then  $G[v_{-1}, v_1]$  is  $(v_{-1}, v_0, v_1)$ -composed.

Let  $r'_2 = r_2 - k$  and  $r'_1 = r_1 - \ell$ .

CLAIM 9.1. *If  $G[v_{-k'}, v_{\ell'}]$  is  $(v_{-k'}, v_0, v_{\ell'})$ -composed with canonical ordering  $v_{-k'}, v_{-k'+1}, \dots, v_{\ell'}$ , then  $k' \leq r'_2 - 1$  and  $\ell' \leq r'_1 - 1$ .*

*Proof.* Let  $D_1, D_2, \dots, D_r$  be a canonical sequence of  $G[v_{-k'}, v_{\ell'}]$  corresponding to the canonical ordering  $v_{-k'}, v_{-k'+1}, \dots, v_{\ell'}$ . Suppose that  $k' > r'_2 - 1$ . Consider the graph  $D' = D_{\widehat{-r'_2}}$ , where  $\widehat{-r'_2}$  is the smallest integer such that  $v_{-r'_2} \in V(D_{\widehat{-r'_2}})$ . Let  $V(D') = [v_{-r'_2}, v_{\ell''}]$ . By Lemma 2.1, there exists a  $(v_0, v_{\ell''})$ -path  $P$  such that  $V(P) = [v_{-r'_2}, u_{\ell''}]$ . Then  $C' = P\overrightarrow{C}[u_{\ell'}, v_{\ell''}]P'R$  is a cycle which contains all the vertices in  $V(C) \cup V(R)$ , a contradiction.  $\square$

In a way similar to Claim 7, we have the next claim.

CLAIM 9.2. *If  $G[v_{-k'}, v_{\ell'}]$  is  $(v_{-k'}, v_0, v_{\ell'})$ -composed with canonical ordering  $v_{-k'}, v_{-k'+1}, \dots, v_{\ell'}$ , where  $k' \leq r'_2 - 1$  and  $\ell' \leq r'_1 - 1$ , and any two nonadjacent vertices in  $[v_{-k'-1}, v_{\ell'+1}]$  have degree sum less than  $n$ , then one of the following is true:*

- (1)  $G[v_{-k'-1}, v_{\ell'}]$  is  $(v_{-k'-1}, v_0, v_{\ell'})$ -composed with canonical ordering  $v_{-k'-1}, v_{-k'}, \dots, v_{\ell'}$ ,
- (2)  $G[v_{-k'}, v_{\ell'+1}]$  is  $(v_{-k'}, v_0, v_{\ell'+1})$ -composed with canonical ordering  $v_{-k'}, v_{-k'+1}, \dots, v_{\ell'+1}$ ,
- (3)  $G[v_{-k'-1}, v_{\ell'+1}]$  is  $(v_{-k'-1}, v_0, v_{\ell'+1})$ -composed with canonical ordering  $v_{-k'-1}, v_{-k'}, \dots, v_{\ell'+1}$ .

Now we choose  $k', \ell'$  such that

- (1)  $G[v_{-k'}, v_{\ell'}]$  is  $(v_{-k'}, v_0, v_{\ell'})$ -composed with canonical ordering  $v_{-k'}, v_{-k'+1}, \dots, v_{\ell'}$ ;
- (2) any two nonadjacent vertices in  $[v_{-k'}, v_{\ell'}]$  have degree sum less than  $n$ ; and
- (3)  $k' + \ell'$  is as big as possible.

In a way similar to Claim 8, we have that  $(v_{-k'-1}, v_{\ell'})$ ,  $(v_{-k'}, v_{\ell'+1})$ , or  $(v_{-k'-1}, v_{\ell'+1})$  is  $v_0$ -good on  $C$ .  $\square$

From Claims 8 and 9, we get that there exists a cycle which contains all the vertices in  $V(C) \cup V(R)$  by Lemma 2.3, a contradiction.

Case 2.  $r = 1$  and  $u_0 v_0 \in E(G)$ . We have  $u_0 u_{-1} \in E(G)$  and  $u_0 u_{-r_2} \notin E(G)$ , where  $u_{-r_2} = v_{-1}$ . Let  $u_{-k-1}$  be the first vertex in  $\overleftarrow{C}[u_{-1}, v_{-1}]$  such that  $u_0 u_{-k-1} \notin E(G)$ . Then  $k \leq r_2 - 1$ .

Similarly, let  $v_{\ell+1}$  be the first vertex in  $\overleftarrow{C}[v_1, u_1]$  such that  $v_0 v_{\ell+1} \notin E(G)$ . Then  $\ell \leq r_1 - 1$ .

CLAIM 10. *Let  $x \in [u_{-k-1}, u_{-1}]$  and  $y \in [v_1, v_{\ell+1}]$ . Then*

- (1)  $xz_1, xv_0 \notin \widetilde{E}(G)$ ,
- (2)  $yz_1, yu_0 \notin \widetilde{E}(G)$ ,
- (3)  $xy \notin \widetilde{E}(G)$ .

*Proof.* (1) If  $x = u_{-1}$ , then by Claims 1 and 3 we have  $u_{-1}z_1, u_{-1}v_0 \notin \tilde{E}(G)$ . So we assume that  $x = u_{-k'}$ , where  $-k' \in [-k - 1, -2]$  and  $u_0u_{-k'+1} \in E(G)$ .

If  $u_{-k'}z_1 \in \tilde{E}(G)$ , then  $C' = u_0u_{-k'+1} \overrightarrow{C}[u_{-k'+1}, u_{-1}]u_{-1}u_1 \overrightarrow{C}[u_1, u_{-k'}]u_{-k'}z_1u_0$  is an  $\alpha$ -cycle which contains all the vertices in  $V(C) \cup V(R)$ , a contradiction.

If  $u_{-k'}v_0 \in \tilde{E}(G)$ , then  $C' = u_0u_{-k'+1} \overrightarrow{C}[u_{-k'+1}, u_{-1}]u_{-1}u_1 \overrightarrow{C}[u_1, v_1]v_1v_{-1} \overrightarrow{C}[v_{-1}, u_{-k'}]u_{-k'}v_0R$  is an  $\alpha$ -cycle which contains all the vertices in  $V(C) \cup V(R)$ , a contradiction.

The assertion (2) can be proved similarly.

(3) If  $x = u_{-1}$  and  $y = v_1$ , then by Claim 3, we have  $xy \notin \tilde{E}(G)$ .

If  $u_{-k'}v_1 \in \tilde{E}(G)$ , where  $k' \in [2, k + 1]$ , then  $C' = u_0R \overrightarrow{C}[v_0, u_{-k'}]u_{-k'}v_1 \overrightarrow{C}[v_1, u_1]u_1u_{-1} \overrightarrow{C}[u_{-1}, u_{-k'+1}]u_{-k'+1}u_0$  is an  $\alpha$ -cycle which contains all the vertices in  $V(C) \cup V(R)$ , a contradiction.

If  $u_{-1}v_{\ell'} \in \tilde{E}(G)$ , where  $\ell' \in [2, \ell + 1]$ , then we can prove the result similarly.

If  $u_{-k'}v_{\ell'} \in \tilde{E}(G)$ , where  $k' \in [2, k + 1]$  and  $\ell' \in [2, \ell + 1]$ , then  $C' = u_0u_{-k'+1} \overrightarrow{C}[u_{-k'+1}, u_{-1}]u_{-1}u_1 \overrightarrow{C}[u_1, v_{\ell'}]v_{\ell'}u_{-k'} \overrightarrow{C}[u_{-k'}, v_{-1}]v_{-1}v_1 \overrightarrow{C}[v_1, v_{\ell'-1}]v_{\ell'-1}v_0R$  is an  $\alpha$ -cycle which contains all the vertices in  $V(C) \cup V(R)$ , a contradiction.  $\square$

CLAIM 11. *Either  $u_{-k-1}u_0 \notin \tilde{E}(G)$  or  $v_{\ell+1}v_0 \notin \tilde{E}(G)$ .*

*Proof.* Assume the opposite. Since  $u_{-k-1}u_0, v_{\ell+1}v_0 \notin E(G)$ , we have  $d(u_{-k-1}) + d(u_0) \geq n$  and  $d(v_{\ell+1}) + d(v_0) \geq n$ . By Claim 10, we have  $d(u_0) + d(v_{\ell+1}) < n$  and  $d(v_0) + d(u_{-k-1}) < n$ , a contradiction.  $\square$

Without loss of generality, we assume that  $u_{-k-1}u_0 \notin \tilde{E}(G)$ . If  $v_{\ell+1}v_0 \notin \tilde{E}(G)$ , then the subgraph induced by  $\{z_1, v_0, v_{\ell}, v_{\ell+1}, u_0, u_{-k}, u_{-k-1}\}$  is a  $D$  which is not heavy, a contradiction. Since  $v_0v_{\ell+1} \notin E(G)$ , we have  $d(v_0) + d(v_{\ell+1}) \geq n$ .

CLAIM 12. *Either  $(v_{-1}, v_1)$  or  $(v_{-1}, v_{\ell+1})$  is  $v_0$ -good on  $C$ .*

*Proof.* If  $v_1v_{-1} \notin E(G)$ , then, by Claim 2,  $d(v_1) + d(v_{-1}) \geq n$ . Then  $P = v_0v_1$  is a  $(v_0, v_1)$ -path and  $D = v_0v_{-1} \cup v_1$  is a  $(v_0v_1, v_{-1}v_1)$ -pair. Thus,  $(v_{-1}, v_1)$  is  $v_0$ -good on  $C$ .

If  $v_1v_{-1} \in E(G)$ , then  $P = v_0v_{\ell} \overrightarrow{C}[v_{\ell}, v_1]v_1v_{-1}$  is a  $(v_0, v_{-1})$ -path and  $D = v_0 \cup v_{-1}v_1 \overrightarrow{C}[v_1, v_{\ell+1}]$  is a  $(v_0v_{-1}, v_0v_{\ell+1})$ -pair. Since  $d(v_0) + d(v_{\ell+1}) \geq n$ , we have that  $(v_{-1}, v_{\ell+1})$  is  $v_0$ -good on  $C$ .  $\square$

CLAIM 13. *If  $G[u_{-k'}, u_{\ell'}]$  is  $(u_{-k'}, u_0, u_{\ell'})$ -composed with canonical ordering  $u_{-k'}, u_{-k'+1}, \dots, u_{\ell'}$ , then  $k' \leq r_2 - 2$  and  $\ell' \leq r_1 - \ell - 1$ .*

*Proof.* The claim can be proved similarly to Claims 6 and 9.1.  $\square$

Now we prove the following claim.

CLAIM 14. *If  $G[u_{-k'}, u_{\ell'}]$  is  $(u_{-k'}, u_0, u_{\ell'})$ -composed with canonical ordering  $u_{-k'}, u_{-k'+1}, \dots, u_{\ell'}$ , where  $k' \leq r_2 - 2$  and  $\ell' \leq r_1 - \ell - 1$ , and any two nonadjacent vertices in  $[u_{-k'-1}, u_{\ell'+1}]$  have degree sum less than  $n$ , then one of the following is true:*

- (1)  $G[u_{-k'-1}, u_{\ell'}]$  is  $(u_{-k'-1}, u_0, u_{\ell'})$ -composed with canonical ordering  $u_{-k'-1}, u_{-k'}, \dots, u_{\ell'}$ ,
- (2)  $G[u_{-k'}, u_{\ell'+1}]$  is  $(u_{-k'}, u_0, u_{\ell'+1})$ -composed with canonical ordering  $u_{-k'}, u_{-k'+1}, \dots, u_{\ell'+1}$ ,
- (3)  $G[u_{-k'-1}, u_{\ell'+1}]$  is  $(u_{-k'-1}, u_0, u_{\ell'+1})$ -composed with canonical ordering  $u_{-k'-1}, u_{-k'}, \dots, u_{\ell'+1}$ .

*Proof.* Assume the opposite, which implies that for every vertex  $u_s \in [u_{-k'+1}, u_{\ell'}]$ ,  $u_{-k'-1}u_s \notin E(G)$ , and for every vertex  $u_s \in [u_{-k'}, u_{\ell'-1}]$ , we have  $u_{\ell'+1}u_s \notin E(G)$  and  $u_{-k'-1}u_{\ell'+1} \notin E(G)$ .

CLAIM 14.1. *Let  $v \in \{v_0, v_1\}$  and  $u_s \in [u_{-k'-1}, u_{\ell'+1}] \setminus \{u_0\}$ . Then  $vu_s \notin \tilde{E}(G)$ .*

*Proof.* In a way similar to Claim 7.1, we have  $v_0u_s \notin \tilde{E}(G)$ .

Now we assume that  $v_1u_s \in \tilde{E}(G)$ . Note that if  $v_0v_2 \notin E(G)$ , then  $d(v_0)+d(v_2) \geq n$ . We have  $v_0v_2 \in \tilde{E}(G)$ .

If  $s \in [-k' - 1, -2]$ , then  $s + 1 \in [-k', -1]$ . By the definition of a composed graph, there exists  $t \in [1, \ell']$  such that  $G[u_{s+1}, u_t]$  is  $(u_{s+1}, u_0, u_t)$ -composed. By Lemma 2.1, there exists a  $(u_0, u_t)$ -path  $P$  such that  $V(P) = [u_{s+1}, u_t]$ . Then  $C' = P\overrightarrow{C}[u_t, v_1]v_1u_s\overleftarrow{C}[u_s, v_0]R$  is an  $o$ -cycle which contains all the vertices in  $V(C) \cup V(R)$ , a contradiction.

If  $s = -1$ , then by Claim 3, we have  $v_1u_{-1} \notin \tilde{E}(G)$ .

If  $s = 1$ , then  $C' = \overleftarrow{C}[u_0, v_{-1}]v_{-1}v_1u_1\overrightarrow{C}[u_1, v_2]v_2v_0R$  is an  $o$ -cycle which contains all the vertices in  $V(C) \cup V(R)$ , a contradiction.

If  $s \in [2, \ell' + 1]$ , then  $s - 1 \in [1, \ell']$ . By the definition of a composed graph, there exists  $t \in [-k', -1]$  such that  $G[u_t, u_{s-1}]$  is  $(u_t, u_0, u_{s-1})$ -composed. By Lemma 2.1, there exists a  $(u_0, u_t)$ -path  $P$  such that  $V(P) = [u_t, u_{s-1}]$ . Then  $C' = P\overrightarrow{C}[u_t, v_{-1}]v_{-1}v_1u_s\overrightarrow{C}[u_s, v_2]v_2v_0R$  is an  $o$ -cycle which contains all the vertices in  $V(C) \cup V(R)$ , a contradiction.  $\square$

Let  $G' = G[[u_{-k'-1}, u_{\ell'}] \cup \{v_0, v_1\}]$  and  $G'' = G[[u_{-k'-1}, u_{\ell'+1}] \cup \{v_0, v_1\}]$ . Then, in a way similar to Claim 7.2, we have the next claim.

CLAIM 14.2.  $G''$ , and then  $G'$ , is  $\{K_{1,3}, N_{1,1,2}\}$ -free.

In a way similar to Claim 7, we can complete the proof of Claim 14.  $\square$

Now we choose  $k', \ell'$  such that

- (1)  $G[v_{-k'}, v_{\ell'}]$  is  $(v_{-k'}, v_0, v_{\ell'})$ -composed with canonical ordering  $v_{-k'}, v_{-k'+1}, \dots, v_{\ell'}$ ;
- (2) any two nonadjacent vertices in  $[v_{-k'}, v_{\ell'}]$  have degree sum less than  $n$ ; and
- (3)  $k' + \ell'$  is as big as possible.

In a way similar to Claim 8, we have the following result.

CLAIM 15.  $(u_{-k'-1}, u_{\ell'})$ ,  $(u_{-k'}, u_{\ell'+1})$ , or  $(u_{-k'-1}, u_{\ell'+1})$  is  $u_0$ -good on  $C$ .

By Claim 13, we have  $k' \leq r_2 - 2$  and  $\ell' \leq r_1 - \ell - 2$ .

From Claims 12 and 15, we can get that there exists a cycle which contains all vertices in  $V(C) \cup V(R)$  by Lemma 2.3, a contradiction.

The proof is complete.  $\square$

**4. Proof of Theorem 1.9.** Let  $C$  be a longest cycle of  $G$ . Set  $n = |V(G)|$  and  $c = |V(C)|$ , and assume that  $G$  is not Hamiltonian; i.e.,  $c < n$ . Then  $V(G) \setminus V(C) \neq \emptyset$ . Since  $G$  is 2-connected, there exists a  $(u_0, v_0)$ -path with length at least 2 which is internally disjoint from  $C$ , where  $u_0, v_0 \in V(C)$ . Let  $R = z_0z_1z_2 \cdots z_{r+1}$ , where  $z_0 = u_0$  and  $z_{r+1} = v_0$ , be such a path, and choose  $R$  as short as possible. Let  $r_1$  and  $r_2$  denote the number of interior vertices in the two subpaths of  $C$  from  $u_0$  to  $v_0$  (note that clearly  $r_1 + r_2 + 2 = c$ ). We specify an orientation of  $C$  and label the vertices of  $C$  using two distinct notations,  $u_i$  and  $v_i$ ,  $-r_2 \leq i \leq r_1$ , such that  $\overrightarrow{C} = u_0u_1u_2 \cdots u_{r_1}v_0u_{-r_2}u_{-r_2+1} \cdots u_{-1}u_0$  and  $\overleftarrow{C} = v_0v_1v_2 \cdots v_{r_1}u_0v_{-r_2}v_{-r_2+1} \cdots v_{-1}v_0$ , where  $u_\ell = v_{r_1+1-\ell}$  and  $u_{-k} = v_{-r_2-1+k}$ . Let  $H$  be the component of  $G - C$  which contains the vertices in  $[z_1, z_r]$ .

As in section 3, we have the following claims.

CLAIM 1. Let  $x \in V(H)$  and  $y \in \{u_1, u_{-1}, v_1, v_{-1}\}$ . Then  $xy \notin \tilde{E}(G)$ .

CLAIM 2.  $u_1u_{-1} \in \tilde{E}(G)$ ,  $v_1v_{-1} \in \tilde{E}(G)$ .

CLAIM 3.  $u_1v_{-1} \notin \tilde{E}(G)$ ,  $u_{-1}v_1 \notin \tilde{E}(G)$ ,  $u_0v_{\pm 1} \notin \tilde{E}(G)$ ,  $u_{\pm 1}v_0 \notin \tilde{E}(G)$ .

CLAIM 4. Either  $u_1u_{-1}$  or  $v_1v_{-1}$  is in  $E(G)$ .

By Claim 4, without loss of generality, we assume that  $u_1u_{-1} \in E(G)$ . Then we have that  $G[u_{-1}, u_1]$  is  $(u_{-1}, u_0, u_1)$ -composed.

CLAIM 5. *If  $G[u_{-k}, u_\ell]$  is  $(u_{-k}, u_0, u_\ell)$ -composed, then  $k \leq r_2 - 2$  and  $\ell \leq r_1 - 2$ .*

The proof of Claim 5 is in a way similar to that of Claim 6 in section 3.

Now we prove the following claim.

CLAIM 6. *If  $G[u_{-k}, u_\ell]$  is  $(u_{-k}, u_0, u_\ell)$ -composed with canonical ordering  $u_{-k}, u_{-k+1}, \dots, u_\ell$ , where  $k \leq r_2 - 2$  and  $\ell \leq r_1 - 2$ , and any two nonadjacent vertices in  $[u_{-k-1}, u_{\ell+1}]$  have degree sum less than  $n$ , then one of the following is true:*

- (1)  $G[u_{-k-1}, u_\ell]$  is  $(u_{-k-1}, u_0, u_\ell)$ -composed with canonical ordering  $u_{-k-1}, u_{-k}, \dots, u_\ell$ ,
- (2)  $G[u_{-k}, u_{\ell+1}]$  is  $(u_{-k}, u_0, u_{\ell+1})$ -composed with canonical ordering  $u_{-k}, u_{-k+1}, \dots, u_{\ell+1}$ ,
- (3)  $G[u_{-k-1}, u_{\ell+1}]$  is  $(u_{-k-1}, u_0, u_{\ell+1})$ -composed with canonical ordering  $u_{-k-1}, u_{-k}, \dots, u_{\ell+1}$ .

*Proof.* Assume the opposite, which implies that for every vertex  $u_s \in [u_{-k+1}, u_\ell]$ ,  $u_{-k-1}u_s \notin E(G)$ , and for every vertex  $u_s \in [u_{-k}, u_{\ell-1}]$ ,  $u_{\ell+1}u_s \notin E(G)$  and  $u_{-k-1}u_{\ell+1} \notin E(G)$ .

CLAIM 6.1. *For every vertex  $z \in \{z_1, z_2\}$  and  $u_s \in [u_{-k-1}, u_{\ell+1}] \setminus \{u_0\}$  we have  $zu_s \notin \tilde{E}(G)$  and if  $z_2u_0 \notin E(G)$ , then also  $z_2u_0 \notin \tilde{E}(G)$ .*

This claim can be proved similarly to Claims 5 and 7.1 in section 3.

Let  $G' = G[[u_{-k-1}, u_\ell] \cup \{z_1, z_2\}]$  and  $G'' = G[[u_{-k-1}, u_{\ell+1}] \cup \{z_1, z_2\}]$ . In a way similar to Claims 7.2 and 7.3 in section 3, we have the next claims.

CLAIM 6.2.  *$G''$ , and then  $G'$ , is  $\{K_{1,3}, N_{1,1,2}, H_{1,1}\}$ -free.*

CLAIM 6.3.  *$N_{G'}(u_0) \setminus \{z_1, z_2\}$  is a clique.*

Now, we define  $N_i = \{x \in V(G') : d_{G'}(x, u_{-k-1}) = i\}$ . Then we have  $N_0 = \{u_{-k-1}\}$ ,  $N_1 = \{u_{-k}\}$ , and  $N_2 = N_{G'}(u_{-k}) \setminus \{u_{-k-1}\}$ .

By the definition of a composed graph, we have that  $|N_2| \geq 2$ . If there are two vertices  $x, x' \in N_2$  such that  $xx' \notin E(G')$ , then the graph induced by  $\{u_{-k}, u_{-k-1}, x, x'\}$  is a claw. Thus  $N_2$  is a clique.

We assume  $u_0 \in N_j$ , where  $j \geq 2$ . Then  $z_1 \in N_{j+1}$  and  $z_2 \in N_{j+1}$  if  $z_2u_0 \in E(G)$ , and  $z_2 \in N_{j+2}$  if  $z_2u_0 \notin E(G)$ .

If  $|N_i| = 1$  for some  $i \in [2, j - 1]$ , say,  $N_i = \{x\}$ , then  $x$  is a cut vertex of the graph  $G[u_{-k}, u_\ell]$ . By the definition of a composed graph,  $G[u_{-k}, u_\ell]$  is 2-connected. This implies  $|N_i| \geq 2$  for every  $i \in [2, j - 1]$ .

CLAIM 6.4. *For  $i \in [1, j]$ ,  $N_i$  is a clique.*

*Proof.* If  $i < j$ , or  $i = j$  and  $z_2u_0 \notin E(G)$ , then we can prove the assertion similarly to Claim 7.4 in section 3. Thus we assume that  $i = j$  and  $z_2u_0 \in E(G)$ .

If  $j = 2$ , the assertion is true by the analysis above. So we assume that  $j \geq 3$ , and we have that  $N_{j-3}, N_{j-2}, N_{j-1}, N_{j+1}$  is nonempty and  $|N_{j-1}| \geq 2$ .

First we prove that for every  $x \in N_j \setminus \{u_0\}$ ,  $u_0x \in E(G)$ . We assume that  $u_0x \notin E(G)$ .

By Claim 6.1 we have  $xz_1 \notin E(G)$ . If  $u_0$  and  $x$  have a common neighbor in  $N_{j-1}$ , denoted  $w$ , then let  $v$  be a neighbor of  $w$  in  $N_{j-2}$ ; but then the graph induced by  $\{w, v, u_0, x\}$  is a claw, a contradiction. Thus we have that  $u_0$  and  $x$  have no common neighbors in  $N_{j-1}$ .

Let  $w$  be a neighbor of  $u_0$  in  $N_{j-1}$ , and  $w'$  be a neighbor of  $x$  in  $N_{j-1}$ . Then  $u_0w', xw \notin E(G)$ . Let  $v$  be a neighbor of  $w$  in  $N_{j-2}$ , and  $u$  be a neighbor of  $v$  in  $N_{j-3}$ . If  $w'v \notin E(G)$ , then the graph induced by  $\{w, v, w', u_0\}$  is a claw, a contradiction.

Thus we have  $w'v \in E(G)$ , and then the graph induced by  $\{v, u, w', x, w, u_0, z_1\}$  is an  $N_{1,1,2}$ , a contradiction.

Thus we have  $u_0x \in E(G)$  for every  $x \in N_j$ . Then, by Claim 6.3, we have that  $N_j$  is a clique.  $\square$

If there exists some vertex  $y \in N_{j+1}$  other than  $z_1$  and  $z_2$ , then we have  $yu_0 \notin E(G)$  by Claim 6.3. Let  $x$  be a neighbor of  $y$  in  $N_j$ ,  $w$  be a neighbor of  $u_0$  in  $N_{j-1}$ , and  $v$  be a neighbor of  $w$  in  $N_{j-2}$ . Then  $xu_0 \in E(G)$  by Claim 6.4, and  $xw \in E(G)$  by Claim 6.3. Thus the graph induced by  $\{w, v, x, y, u_0, z_1, z_2\}$  is an  $N_{1,1,2}$  if  $z_2u_0 \notin E(G)$ , and is an  $H_{1,1}$  if  $z_2u_0 \in E(G)$ , a contradiction. So we assume that all vertices in  $[u_{-k}, u_\ell]$  are in  $\bigcup_{i=1}^j N_i$ .

If  $u_\ell \in N_j$ , then let  $w$  be a neighbor of  $u_0$  in  $N_{j-1}$ , and  $v$  be a neighbor of  $w$  in  $N_{j-2}$ . Then the graph induced by  $\{w, v, u_0, z_1, u_\ell, u_{\ell+1}\}$  is an  $N_{1,1,2}$  if  $z_2u_0 \notin E(G)$ , and is an  $H_{1,1}$  if  $z_2u_0 \in E(G)$ , a contradiction. Thus we have that  $u_\ell \notin N_j$  and then  $j \geq 3$ .

Let  $u_\ell \in N_i$ , where  $i \in [2, j - 1]$ . If  $u_\ell$  has a neighbor in  $N_{i+1}$ , then let  $y$  be a neighbor of  $u_\ell$  in  $N_{i+1}$ , and  $w$  be a neighbor of  $u_\ell$  in  $N_{i-1}$ . Then the graph induced by  $\{u_\ell, w, y, u_{\ell+1}\}$  is a claw, a contradiction. Thus we have that  $u_\ell$  has no neighbors in  $N_{i+1}$ .

Let  $x \in N_i$  be a vertex other than  $u_\ell$  which has a neighbor  $y$  in  $N_{i+1}$  such that it has a neighbor  $z$  in  $N_{i+2}$ . Let  $w$  be a neighbor of  $x$  in  $N_{i-1}$ , and  $v$  be a neighbor of  $w$  in  $N_{i-2}$ . If  $u_\ell w \notin E(G)$ , then the graph induced by  $\{x, w, u_\ell, y\}$  is a claw, a contradiction. Thus we have that  $u_\ell w \in E(G)$ . Then the graph induced by  $\{w, v, u_\ell, u_{\ell+1}, x, y, z\}$  is an  $N_{1,1,2}$ , a contradiction.

Thus the claim holds.  $\square$

Now we choose  $k, \ell$  such that

- (1)  $G[u_{-k}, u_\ell]$  is  $(u_{-k}, u_0, u_\ell)$ -composed with canonical ordering  $u_{-k}, u_{-k+1}, \dots, u_\ell$ ;
- (2) any two nonadjacent vertices in  $[u_{-k}, u_\ell]$  have degree sum less than  $n$ ; and
- (3)  $k + \ell$  is as big as possible.

In a way similar to Claims 8 and 9 in section 3, we have the next two results.

CLAIM 7.  $(u_{-k-1}, u_\ell)$ ,  $(u_{-k}, u_{\ell+1})$ , or  $(u_{-k-1}, u_{\ell+1})$  is  $u_0$ -good on  $C$ .

CLAIM 8. There exist  $v_{-k'} \in V(\vec{C}[v_{-1}, u_{-k-1}])$  and  $v_{\ell'} \in V(\overleftarrow{C}[v_1, u_{\ell+1}])$  such that  $(v_{-k'}, v_{\ell'})$  is  $v_0$ -good on  $C$ .

From Claims 7 and 8, we can get that there exists a cycle which contains all the vertices in  $V(C) \cup V(R)$  by Lemma 2.3, a contradiction.

The proof is complete.  $\square$

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