

Stability of hereditary graph classes under closure operations

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Abstract

If \mathcal{C} is a subclass of the class of claw-free graphs, then \mathcal{C} is said to be stable if, for any $G \in \mathcal{C}$, the local completion of G at any vertex is also in \mathcal{C} . If cl is a closure operation that turns a claw-free graph into a line graph by a series of local completions and \mathcal{C} is stable, then $\text{cl}(G) \in \mathcal{C}$ for any $G \in \mathcal{C}$. In this paper we study stability of hereditary classes of claw-free graphs defined in terms of a family of connected closed forbidden subgraphs. We characterize line graph preimages of graphs in families that yield stable classes, we identify minimal families that yield stable classes in the finite case, and we also give a general background for techniques for handling unstable classes by proving that their closure may be included into another (possibly stable) class.

1 Introduction

Bedrossian [1] and Faudree and Gould [9] characterized pairs of connected graphs X, Y such that an (X, Y) -free graph G is hamiltonian if and only if G is 2-connected. Brousek et al. [7] characterized connected graphs Y such that the class of $(K_{1,3}, Y)$ -free graphs is stable under the closure operation for hamiltonicity introduced by the third author [11]. Comparing these characterizations (see Theorems A and C), it can be observed that some classes have both properties, but there are also connected graphs Y such that every 2-connected $(K_{1,3}, Y)$ -free graph is hamiltonian while the class of $(K_{1,3}, Y)$ -free graphs is

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not stable. This is particularly the case with $Y = B_{1,2}$ (see Fig. 1) while, e.g., in the class of $(K_{1,3}, N_{1,1,1})$ -free graphs 2-connectedness implies hamiltonicity and the class is stable. This “uneven” behavior was explained in [12], where it was shown that the closure of a 2-connected $(K_{1,3}, B_{1,2})$ -free graph must be $(K_{1,3}, N_{1,1,1})$ -free, i.e.,

$$\text{cl}(\text{Forb}_2(K_{1,3}, B_{1,2})) \subset \text{Forb}_2(K_{1,3}, N_{1,1,1}) \quad (*)$$

(where $\text{Forb}_k(\mathcal{X})$ denotes the class of all k -connected \mathcal{X} -free graphs), and thus, after applying the closure operation, the class of $(K_{1,3}, B_{1,2})$ -free graphs becomes redundant. The technique of the paper [12] was recently extended in [13] for some other graph classes and their behavior under the closure for 2-factors introduced in [16].

In this paper, we develop a general technique for proving inclusions of classes similar to that of (*) for various closure operations, which gives a tool for handling unstable classes using closure techniques.

It should be noted that the results of the papers [12] and [13], which motivated our research, have complicated and technical proofs. Thus, the relative simplicity of the proofs of our results, giving a common generalization, should be considered as another demonstration of the power of closure techniques.

By a *graph* we will always mean a finite simple undirected graph $G = (V(G), E(G))$; whenever we allow multiple edges we say that G is a *multigraph*. Throughout the paper, we assume all graphs in consideration to be connected. We follow the most common graph-theoretical terminology and for concepts and notations not defined here we refer e.g. to [3].

Specifically, $N_G(x)$ denotes the *neighborhood* and $d_G(x)$ the *degree* of a vertex x in G , and for $k \geq 0$ we set $V_k(G) = \{x \in V(G) \mid d_G(x) = k\}$ and $V_{\geq k}(G) = \{x \in V(G) \mid d_G(x) \geq k\}$. A *pendant edge* is an edge with one vertex of degree 1, and a *cherry* in G is a pair of pendant edges sharing a vertex. If G has no cherry, we say G is *cherry-free*. A *clique* in G is a (not necessarily maximal) complete subgraph of G ; a vertex $x \in V(G)$ is *simplicial* if $N_G(x)$ is a clique and *universal* if $N_G(x) = V(G) \setminus \{x\}$. For $x \in V(G)$ we further set $E(x) = \{e \in E(G) \mid e \text{ contains } x\}$. For $X \subset V(G)$, $\langle X \rangle$ denotes the *induced subgraph* on X , and if F is an induced subgraph of G , we also write $F \stackrel{\text{IND}}{\subset} G$. Throughout the paper, $\Delta(G)$ denotes the *maximum degree* of G , $c(G)$ the *circumference* of G , $g(G)$ the *girth* of G and $\nu(G)$ the *cyclomatic number* $\nu(G) = |E(G)| - |V(G)| + 1$ of G .

A hamiltonian path (cycle) in G is a path (cycle) containing all vertices of G . A graph with a hamiltonian cycle is said to be *hamiltonian*, while a graph G is said to be *Hamilton-connected* if G has a hamiltonian (a, b) -path for any $a, b \in V(G)$. Recall that a hamiltonian graph must be 2-connected and a Hamilton-connected graph must be 3-connected.

If \mathcal{X} is a family of graphs, we say that a graph G is *\mathcal{X} -free* if G does not contain any graph from \mathcal{X} as an induced subgraph. The class of all \mathcal{X} -free graphs is denoted $\text{Forb}(\mathcal{X})$. If \mathcal{X} is finite and $\mathcal{X} = \{X_1, \dots, X_k\}$, then we also say that G is (X_1, \dots, X_k) -free and we write $G \in \text{Forb}(X_1, \dots, X_k)$. In this context, the graphs in \mathcal{X} will be referred to as *forbidden induced subgraphs*. Specifically, the graph $C = K_{1,3}$

is called the *claw* and graphs in $\text{Forb}(C)$ are said to be *claw-free*. For $k \geq 1$, we set $\text{Forb}_k(\mathcal{X}) = \{X \in \text{Forb}(\mathcal{X}) \mid x \text{ is } k\text{-connected}\}$. Throughout, P_i denotes the path on i vertices and T denotes the triangle. The graph $K_4 - e$ is called the *diamond*. Further graphs often used as forbidden induced subgraphs are shown in Fig. 1; here the graph $B_{i,j}$

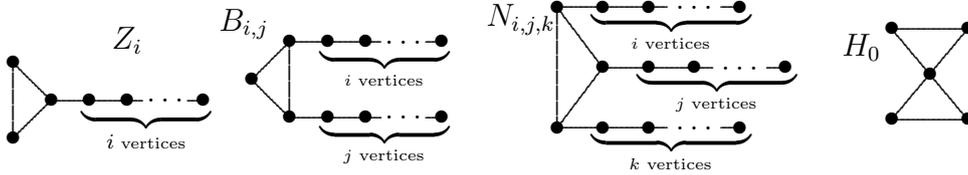


Figure 1

is called the *generalized bull*, $N_{i,j,k}$ the *generalized net* and H_0 the *hourglass*.

We write $G = L(H)$ if G is the *line graph* of H , and in this context we allow H to be a multigraph. For a class of graphs \mathcal{C} we denote $L(\mathcal{C}) = \{L(G) \mid G \in \mathcal{C}\}$.

It is a well-known fact that if G is a line graph (of a graph) and G is not a triangle, then there is a unique graph H such that $G = L(H)$. Such a graph H is called the *preimage* of G and denoted $H = L^{-1}(G)$. We will use a similar notation, i.e. $x = L(y)$ and $y = L^{-1}(x)$, also for edges in H and the corresponding vertices in G . Recall that a graph F is a subgraph of a graph H (we write $F \subset H$) if and only if $L(F) \overset{\text{IND}}{\subset} L(H)$, and $G = L(H)$ is k -connected if and only if H is essentially k -edge-connected (i.e., $|E(H)| \geq k + 1$ and every edge-cut of G separating two nontrivial components has at least k edges).

2 Preliminary results

Pairs of connected graphs X_1, X_2 implying that a 2-connected (X_1, X_2) -free graph is hamiltonian were characterized by Bedrossian [1]. Faudree and Gould [9] reconsidered the Bedrossian characterization and added Z_3 to the list under the additional assumption $n \geq 10$ (where the 'only if' part is now based on infinite families of graphs).

Theorem A [1], [9]. *Let X_1, X_2 be connected graphs with $X_1, X_2 \not\cong P_3$ and let G be a 2-connected graph of order $n \geq 10$ that is not a cycle. Then, G being (X_1, X_2) -free implies G is hamiltonian if and only if (up to a symmetry) $X_1 = K_{1,3}$ and X_2 is an induced subgraph of at least one of the graphs $P_6, Z_3, B_{1,2}$ or $N_{1,1,1}$.*

The third author introduced a closure concept in the class of claw-free graphs as follows (see [11] or also the survey paper [5]). For $x \in V(G)$, the *local completion of G at x* is the graph $G_x^* = (V(G), E(G) \cup \{uv \mid u, v \in N_G(x)\})$ (i.e., G_x^* is obtained from G by adding to $\langle N_G(x) \rangle_G$ all missing edges); the vertex x is called the *center* of the completion in this context. A vertex $x \in V(G)$ is said to be *eligible* if $\langle N_G(x) \rangle_G$ is a connected noncomplete graph. The *closure* of a claw-free graph G is the graph $\text{cl}(G)$ obtained from G by recursively performing the local completion operation at eligible vertices, as long as

this is possible (more precisely, there is a sequence of graphs G_1, \dots, G_k such that $G_1 = G$, $G_{i+1} = (G_i)_x^*$, for some vertex $x \in V(G)$ eligible in G_i , $i = 1, \dots, k-1$, and $G_k = \text{cl}(G)$). We say that G is *closed* if $G = \text{cl}(G)$. The following result summarizes basic properties of the closure operation.

Theorem B [11]. *Let G be a claw-free graph. Then*

- (i) $\text{cl}(G)$ is uniquely determined,
- (ii) $c(\text{cl}(G)) = c(G)$,
- (iii) $\text{cl}(G)$ is the line graph of a triangle-free graph.

Thus, the closure operation turns a claw-free graph G into a unique line graph of a triangle-free graph while preserving the length of a longest cycle (and hence also the hamiltonicity or nonhamiltonicity) of G .

The behavior of some further path and cycle properties under the closure has also been studied. It turns out that the closure operation preserves some of these properties (e.g., the existence of a 2-factor - see [14]), while some other properties (such as e.g. the Hamilton-connectedness, see [2]) are not necessarily preserved. Consequently, several further closure concepts have been developed – see e.g., [4], [15] for strengthening of the closure $\text{cl}(G)$ for hamiltonicity, [16] for a closure for 2-factors, or [17] for a closure for Hamilton-connectedness. All these closure concepts are based on the local completion operation, i.e., the closure of a graph G is obtained by a series of local completions. The difference is in the definition of eligibility. Further details are omitted since they are not needed for our results; they are well documented in the references mentioned above.

Let \mathcal{C} be a subclass of the class of claw-free graphs. We say that \mathcal{C} is *stable* if $G \in \mathcal{C}$ implies $G_x^* \in \mathcal{C}$, for any $x \in V(G)$. Note that if cl is any closure operation based on local completions (i.e., such that $\text{cl}(G)$ is obtained from G by a series of local completions), then a class \mathcal{C} being stable implies $\text{cl}(G) \in \mathcal{C}$, for any $G \in \mathcal{C}$.

An example of a stable class is the class $\text{Forb}(C, P_i)$ of all (C, P_i) -free graphs for any fixed $i \geq 3$, or the class $\text{Forb}(C, N_{i,j,k})$ for any $i, j, k \geq 1$ (see [7]). An example of a stable class defined in terms of a larger (but finite) family of forbidden subgraphs is the class $\text{Forb}(C, S_1, S_2, N_{1,1,2})$, where S_1, S_2 are the graphs shown in Fig. 2. Proof of the fact that

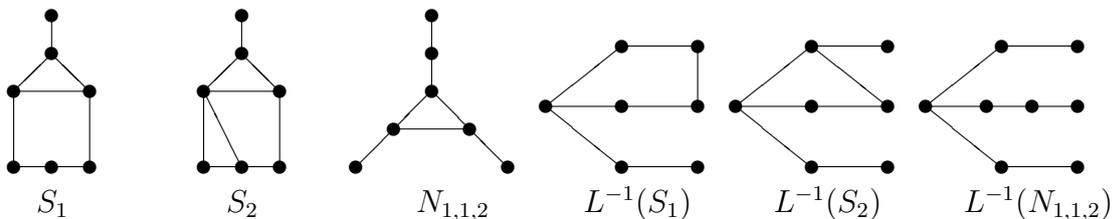


Figure 2

$\text{Forb}(C, S_1, S_2, N_{1,1,2})$ is stable is implicit in the proof of Proposition 3 of [12] (Claims 1,

2 and 3). The class $\text{Forb}(C, \{C_k\}_{k=k_0}^\infty)$, where $k_0 \geq 3$ is any fixed integer, is an example of a stable class with infinite family of forbidden subgraphs (for stability proof see [8], Lemma 2).

Note that all these classes are stable without any assumption on the eligibility of the vertices used as centers of local completions.

On the other hand, the graph G_1 in Fig. 3 is an example of a graph that is (C, H_0) -free while $(G_1)_x^*$ contains an induced H_0 . A similar example for the case of forbidden subgraph

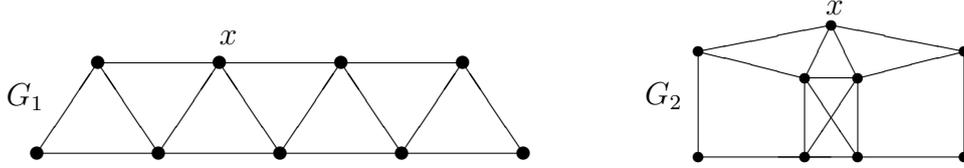


Figure 3

Z_3 is the graph G_2 in Fig. 3. Thus, the graphs in Fig. 3 show that the classes $\text{Forb}(C, H_0)$ and $\text{Forb}(C, Z_3)$ are not stable. However, continuing with performing local completions and using the eligibility assumption at their centers, it is still possible to show that $\text{cl}(G_1)$ is (C, H_0) -free and $\text{cl}(G_2)$ is (C, Z_3) -free.

We say that a class \mathcal{C} is *weakly stable under a closure operation* cl if $\text{cl}(G) \in \mathcal{C}$ for any $G \in \mathcal{C}$. It was shown in [6] that $\text{Forb}(C, Z_i)$ is weakly stable under cl for any $i \geq 1$ and, similarly, $\text{Forb}(C, H_0)$ is also weakly stable under cl (see [7]).

Connected graphs X for which $\text{Forb}(C, X)$ is weakly stable under cl were characterized in [7].

Theorem C [7]. *Let X be a closed connected claw-free graph. Then $\text{Forb}(C, X)$ is weakly stable under cl if and only if*

$$X \in \{H_0, T\} \cup \{P_i \mid i \geq 3\} \cup \{Z_i \mid i \geq 1\} \cup \{N_{i,j,k} \mid i, j, k \geq 1\}.$$

From Theorem C and the previous examples we easily conclude that if X is a closed connected claw-free graph then the class $\text{Forb}(C, X)$ is stable if and only if X is a path or a generalized net.

Note that the situation can be different with different types of closures that use a different eligibility concept: e.g., the class $\text{Forb}(C, Z_i)$, which is weakly stable under cl by Theorem C, is not weakly stable under the multigraph closure for Hamilton-connectedness $\text{cl}^M(G)$ introduced in [17].

Comparing Theorems A and C, we observe a somewhat surprising fact that, although in the class $\text{Forb}(C, B_{1,2})$ 2-connectedness implies hamiltonicity, $\text{Forb}(C, B_{1,2})$ is not weakly stable under cl . This is explained by the fact that, although the classes $\text{Forb}(C, B_{1,2})$ and $\text{Forb}(C, N_{1,1,1})$ are independent, the closures of graphs from $\text{Forb}(C, B_{1,2})$ are in $\text{Forb}(C, N_{1,1,1})$. Specifically, the following was proved in [12] where $\text{cl}_C(G)$ is the cycle closure of G , as introduced in [4].

Theorem D [12]. *Let G be a 2-connected graph.*

- (i) *If G is (C, P_6) -free, then $\text{cl}(G)$ is $(C, N_{1,1,1})$ -free.*
- (ii) *If G is $(C, B_{1,2})$ -free, then $\text{cl}(G)$ is $(C, N_{1,1,1})$ -free.*
- (iii) *If G is (C, Z_3) -free, then $\text{cl}_C(G)$ is $(C, N_{1,1,1})$ -free.*

Corollary E [12]. *Let G be a 2-connected (X_1, X_2) -free graph of order $n \geq 11$, where X_1, X_2 is a pair of connected graphs such that G being (X_1, X_2) -free implies G is hamiltonian. Then G is claw-free and $\text{cl}_C(G)$ is $N_{1,1,1}$ -free.*

Thus, $\text{Forb}(C, N_{1,1,1})$ is the only class that comes into consideration when working with closed graphs. Similar results were obtained in [13], where the characterization of forbidden pairs for 2-factors from [8] was simplified using the 2-factor closure introduced in [16].

In this paper we will shed more light on the stability of hereditary classes of type $\text{Forb}(C, \mathcal{M})$, where \mathcal{M} can be an arbitrary (even infinite) family of connected closed claw-free graphs, and we will identify some common background for the techniques used in proving results similar to Theorem D. As we will be interested in using the correspondence between a line graph and its preimage, and all forbidden subgraphs in the previous results are closed, we restrict our considerations to the cases when all forbidden subgraphs (besides the claw) are closed claw-free graphs.

3 Stable classes

We begin with some necessary definitions. The following concepts slightly extend those introduced in [13].

Let G be a graph, $x \in V(G)$ a vertex of degree at least 2, and let A, B be a partition of $E(x)$ (i.e., $E(x) = A \cup B$, where A, B are disjoint and nonempty). Let G_x^+ be the graph with $V(G_x^+) = (V(G) \setminus \{x\}) \cup \{x_1, x_2\}$, where $x_1, x_2 \notin V(G)$ (i.e., x_1 and x_2 are “new” vertices), in which $x_1x_2 \in E(G_x^+)$, and for every edge $e \in E(G)$ with vertices u, x the graph G_x^+ contains an edge from u to x_1 if $e \in A$, or from u to x_2 if $e \in B$, respectively. We say that G_x^+ is obtained from G by *splitting of type 1 of the vertex x* .

Let $e \in E(G)$ be a pendant edge with vertices u, x , $d_G(x) \geq 3$. We say that a graph $G_x^{+(e)}$ is obtained from G by *splitting of type 2 of x* , if $G_x^{+(e)}$ is obtained from the graph $(V(G) \setminus \{u\}, E(G) \setminus \{e\})$ by splitting of type 1 of x (see Fig. 4).

The following result, generalizing Proposition 1 of [13], is the first step towards identifying the line graph preimage counterpart of stability.

Proposition 1. *Let G be a claw-free graph and $x \in V(G)$. If G_x^* contains a connected induced subgraph X such that $X = L(Y)$, where Y is a triangle-free and cherry-free graph, then G contains an induced subgraph X' such that $X' = L(Y')$, where either Y, Y' are isomorphic, or Y' is obtained from Y by splitting of a vertex.*



Figure 4

Proof. Let $X \stackrel{\text{IND}}{\subset} G_x^*$ and $Y = L^{-1}(X)$ be graphs satisfying the assumptions and note that the assumption that Y is triangle-free implies that X is diamond-free. Set $B = E(X) \setminus E(G)$ and $S = E(X) \cap E(G)$. If $B = \emptyset$, then we set $X' = X$ and we are done, hence suppose $B \neq \emptyset$.

Since X is induced, all edges in B are in one clique of X ; thus, let K be a largest clique containing B .

Claim 1. *Let $u, w_1, w_2 \in V(K)$ be such that $uw_1, uw_2 \in S$ but $w_1w_2 \in B$. Then $N_X(u) = V(K) \setminus \{u\}$. In particular, u is simplicial in X .*

Proof. Suppose that, to the contrary, $uv \in E(X)$ for some $v \in V(X) \setminus V(K)$. Since $\langle \{u, v, w_1, w_2\} \rangle_G \not\cong K_{1,3}$, up to a symmetry, $vw_1 \in E(G)$. Since X is induced, $vw_1 \in E(X)$. Let $z \in V(K)$ be arbitrary (not excluding the possibility $z = w_2$). Since X is diamond-free, $\langle \{u, w_1, v, z\} \rangle_X$ cannot be a diamond (note that possibly uz and w_1z can be in B or in S), implying $vz \in E(X)$. Since z is arbitrary, v is adjacent (in X) to all vertices of K , contradicting the maximality of K . \square

Claim 2. *At most one vertex in K is simplicial in X .*

Proof. Since Y is triangle-free, the vertices of K correspond to a star in Y . Two simplicial vertices in K would give a cherry in Y , a contradiction. \square

Let K_S be the graph with $V(K_S) = V(K)$ and $E(K_S) = S$ and, similarly, set $K_B = (V(K), B)$. Note that K_B is triangle-free for, otherwise, x would be a claw center in G .

Claim 3. *The graph K_S satisfies one of the following:*

- (i) K_S has two components and both are cliques,
- (ii) K_S contains a vertex u which is simplicial in X , the graph $K_S - u$ has two complete components, and u is universal in K_S .

Proof. Suppose first that no vertex of K is simplicial (in X). By Claim 1, each component of K_S is a clique. If K_S has one component, then $B = \emptyset$, a contradiction. If K_S has at least three components, then, for any three vertices a_1, a_2, a_3 in different components, $\langle \{a_1, a_2, a_3\} \rangle_X$ is a triangle in B , a contradiction again. Hence K_S has two components and K_S satisfies (i).

Let now $u \in V(K)$ be simplicial in X and let A_1, \dots, A_k be components of $K_S - u$. First observe that all A_i are cliques, for otherwise, by Claim 1, we have in K a second vertex that is simplicial in X , contradicting Claim 2.

If $k \geq 3$, then, as above, we have a triangle in B , a contradiction. Next, suppose that $k = 1$. If all edges between u and A_1 are in B , K_S satisfies (i) and we are done. Thus, let $v \in V(A_1)$ be a neighbor of u such that $uv \in S$. If there is a $w \in V(A_1)$ such that $uw \in B$, then, by Claim 1, v is a second simplicial vertex, a contradiction. Thus, all edges between u and A_1 are in S , but then $B = \emptyset$, a contradiction again. Hence we have $k = 2$. If all edges between u and A_1, A_2 are in B , then, for some $a_i \in V(A_i)$, $i = 1, 2$, $\langle \{u, a_1, a_2\} \rangle_X$ is a triangle in K_B , a contradiction. Hence we can choose notation such that $ua_1 \in E(K_S)$. By Claims 1 and 2, u is adjacent in K_S to all vertices in A_1 , i.e., $S_1 = \langle V(A_1) \cup \{u\} \rangle_{K_S}$ is a clique.

If u is adjacent in K_S to no vertex of A_2 , then S_1 and $S_2 = A_2$ are complete components of K_S and K_S satisfies (i). Thus, suppose that $ua_2 \in E(G)$ (i.e. $ua_2 \in S$) for some $a_2 \in A_2$. By Claims 1 and 2, u is adjacent in K_S to all vertices of A_2 and K_S satisfies (ii). \square

Now we can finish the proof of Proposition 1. We distinguish the two cases given in Claim 3.

(i) The graph K_S has two complete components S_1, S_2 (one of them possibly containing a simplicial vertex u). Since K_S is disconnected, x is adjacent to all vertices of K and, by the maximality of K , $x \notin V(X)$. We set $X' = \langle V(X) \cup \{x\} \rangle_G$. Since $N_{X'}(x)$ consists of two cliques, X' is a line graph, and in $Y' = L^{-1}(X')$ the contraction of the edge $y = L^{-1}(x)$ yields the graph $Y = L^{-1}(X)$. Hence Y' is obtained from Y by vertex splitting of type 1.

(ii) The vertex u is simplicial in X and $K_S - u$ has two complete components S_1, S_2 . Then we simply set $X' = \langle V(X) \rangle_G$. Since $N_{X'}(u)$ consists of two cliques, X' is a line graph and, similarly as above, in $Y' = L^{-1}(X')$ the contraction of the edge $y = L^{-1}(u)$ and adding a pendant edge to the contracted vertex yields the graph $Y = L^{-1}(X)$. Hence Y' is obtained from Y by vertex splitting of type 2. \blacksquare

Examples.

1. Let X, X', Y be the graphs in Fig. 5. If G contains an induced copy of X' , then

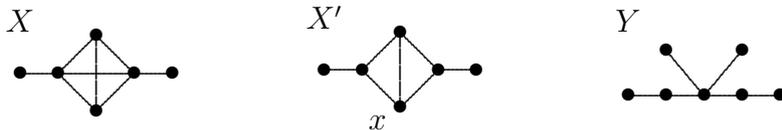


Figure 5

G_x^* contains an induced copy of X . We have $X = L(Y)$, however, there is no graph Y' of Proposition 1 since X' is not a line graph. This example shows that the assumption that Y is cherry-free is necessary in Proposition 1.

2. Similarly, for the graphs X , X' , Y in Fig. 6, clearly $X = (X')_x^*$ and $L(Y) = X$, but X' (and also $X' - x$) is not a line graph. Hence also the assumption that Y is triangle-free cannot be omitted in Proposition 1.

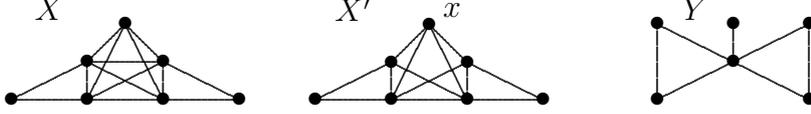


Figure 6

Let \mathcal{Y} be a family of graphs. We say that \mathcal{Y} is *closed under vertex splitting* (or briefly *split-closed*), if, for any $Y \in \mathcal{Y}$ and any Y' obtained from Y by vertex splitting (of type 1 or 2), Y' contains a subgraph $Y'' \in \mathcal{Y}$.

The following result characterizes families of closed forbidden subgraphs that yield stable classes.

Theorem 2. *Let \mathcal{H} be the class of all connected triangle-free and cherry-free graphs and let $\mathcal{Y} \subset \mathcal{H}$ and $\mathcal{X} = L(\mathcal{Y})$. Then the class $\text{Forb}(C, \mathcal{X})$ is stable if and only if \mathcal{Y} is split-closed.*

Proof. If \mathcal{Y} is split-closed, then the class $\text{Forb}(C, \mathcal{X})$ is stable by Proposition 1.

Conversely, suppose that \mathcal{Y} is not split-closed. Then there is an $H \in \mathcal{Y}$, a $y \in V(H)$ and a partition of edges at y such that the graph H' , obtained from H by splitting at y , contains no subgraph from \mathcal{Y} . Set $G = L(H')$, let $h \in E(H')$ be the new edge obtained by splitting y , and set $x = L(h) \in V(G)$. Since H' contains no subgraph from \mathcal{Y} , the graph $G = L(H')$ contains no induced subgraph from $\mathcal{X} = L(\mathcal{Y})$, hence $G \in \text{Forb}(C, \mathcal{X})$. However, $G_x^* = L(H)$ and $H \in \mathcal{Y}$, hence $L(H) \in \mathcal{X} = L(\mathcal{Y})$ and $G_x^* \notin \text{Forb}(C, \mathcal{X})$. Thus, $\text{Forb}(C, \mathcal{X})$ is not stable. ■

Examples.

1. The classes $\text{Forb}(C, \mathcal{X})$, where $\mathcal{X} = \{P_i\}$ for any fixed $i \geq 2$ or $\mathcal{X} = \{N_{i,j,k}\}$ for any fixed $i, j, k \geq 1$ are stable, and it is straightforward to see that any path or any preimage of a generalized net is a subgraph of any graph obtained from it by vertex splitting.

2. Consider the graphs S_1 , S_2 and $N_{1,1,2}$ of Fig. 2. It is easy to see that splitting of any vertex in any of the graphs $L^{-1}(S_1)$, $L^{-1}(S_2)$ or $L^{-1}(N_{1,1,2})$ gives a graph containing $L^{-1}(N_{1,1,2})$, hence $\text{Forb}(C, S_1, S_2, N_{1,1,2})$ is stable.

3. Similarly, splitting of any vertex in a cycle of length i gives a cycle of length $i + 1$, hence $\text{Forb}(C, \{C_i\}_{i=k}^\infty)$ is stable for any fixed $k \geq 3$.

4. If Y is any of the graphs $L^{-1}(H_0)$, $L^{-1}(Z_i)$ or $L^{-1}(B_{i,j})$ (see Fig. 7), then splitting of type 2 of its vertex of degree 3 yields a graph not containing Y as a subgraph. Hence none of the classes $\text{Forb}(C, H_0)$, $\text{Forb}(C, Z_i)$, $\text{Forb}(C, B_{i,j})$ is stable.

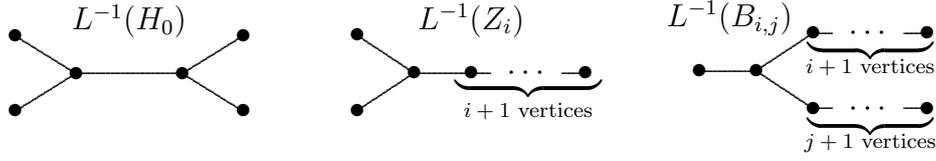


Figure 7

4 Minimal families of forbidden subgraphs

By Theorem 2, $\text{Forb}(C, S_1, S_2, N_{1,1,2})$, where $S_1, S_2, N_{1,1,2}$ are the graphs in Figure 2, is a stable class (see also Example 2 after Theorem 2). However, while neither $\text{Forb}(C, S_1)$ nor $\text{Forb}(C, S_2)$ is stable, $\text{Forb}(C, N_{1,1,2})$ is stable. Thus, the family $\{S_1, S_2, N_{1,1,2}\}$ contains a graph (in this case $N_{1,1,2}$) that itself yields a stable class. As a consequence of the following result we will see that, in the case of a finite family of forbidden subgraphs for a stable class, one of the forbidden subgraphs must always be a path or a generalized net.

We say that a graph Y is a *subdivision of a star*, if Y can be obtained from a star $K_{1,r}$ (for some $r \geq 3$) by adding at least one vertex of degree 2 to each of its edges. In the special case $r = 3$ we say that Y is a *subdivision of the claw*. Note that Y is a subdivision of the claw if and only if $X = L(Y)$ is a generalized net.

Proposition 3. *Let $\mathcal{Y} = \{Y_1, \dots, Y_k\}$, $k \geq 1$, be a split-closed family of connected graphs. Then some $Y_i \in \mathcal{Y}$ is a path or a subdivision of the claw.*

Proof. Let $\mathcal{Y} = \{Y_1, \dots, Y_k\}$, $k \geq 1$, be split-closed.

Claim 1. *If \mathcal{Y} contains a graph Y with $\nu(Y) = r \geq 1$, then \mathcal{Y} contains a graph Y' with $\nu(Y') \leq r - 1$.*

Proof. Let $e = u_1u_2 \in E(Y)$, and let $\{Y_j\}_{j=1}^{\infty}$ be the sequence of graphs in which Y_j is obtained by subdividing the edge e j -times, $j = 1, 2, \dots$. Since \mathcal{Y} is split-closed, each Y_j contains a subgraph from \mathcal{Y} , and since \mathcal{Y} is finite, all graphs in some infinite subsequence of $\{Y_j\}_{j=1}^{\infty}$ have a common subgraph from \mathcal{Y} . But the only graph that can be a common subgraph for any infinite subsequence of $\{Y_j\}_{j=1}^{\infty}$ is a graph obtained from a subgraph of $Y - e$ by possibly adding a path to each of the vertices u_1, u_2 , which has cyclomatic number at most $r - 1$. \square

Claim 2. *The class \mathcal{Y} contains a tree.*

Proof follows immediately from Claim 1 by induction. \square

Claim 3. *The class \mathcal{Y} contains a path or a subdivision of a star.*

Proof. If \mathcal{Y} contains a tree T such that $|V_{\geq 3}(T - V_1(T))| \leq 1$, i.e., $T - V_1(T)$ is a path or a subdivision of a star, we are done since then a subdivision of $T - V_1(T)$ (which is also a path or a subdivision of a star) can be obtained from T by a sequence of vertex splittings of type 2. Hence suppose that $|V_{\geq 3}(T - V_1(T))| \geq 2$ for every tree $T \in \mathcal{Y}$. Let $T_0 \in \mathcal{Y}$ be such that $|V_{\geq 3}(T_0 - V_1(T_0))|$ is minimum and let $u_1, u_2 \in V(T_0)$ be of degree at least 3 in $T_0 - V_1(T_0)$. Let $e \in E(T_0)$ be an edge of the (only) (u_1, u_2) -path in T and let $\{Y_j\}_{j=1}^{\infty}$ be the sequence of graphs obtained by subdividing the edge e j -times, $j = 1, 2, \dots$. Since \mathcal{Y} is split-closed and finite, there is an infinite subsequence of $\{Y_j\}_{j=1}^{\infty}$ with a common subgraph $T_1 \in \mathcal{Y}$. Since every graph in \mathcal{Y} is connected, this is possible only if T_1 is obtained from a component of $T_0 - e$ by possibly adding a path to the endvertex of e . But then $|V_{\geq 3}(T_1 - V_1(T_1))| < |V_{\geq 3}(T_0 - V_1(T_0))|$, contradicting the choice of T_0 . \square

Now let $T \in \mathcal{Y}$ be a path or a subdivision of a star such that $d = \Delta(T)$ is minimum. If $2 \leq d \leq 3$ we are done, hence suppose $d \geq 4$, and let $x \in V(T)$ be of degree d in T . Let A, B be a partition of $E(x)$ with $\min\{|A|, |B|\} \geq 2$ and let T_1 be obtained from T by splitting x with respect to the partition A, B . Then $\Delta(T_1) < \Delta(T)$.

Let $\{Y_j\}_{j=1}^{\infty}$ be the sequence of graphs obtained from T_1 by subdividing j -times, $j = 1, 2, \dots$, the edge x_1x_2 (where x_1, x_2 are the vertices obtained by splitting x). Then for any infinite subsequence of $\{Y_j\}_{j=1}^{\infty}$ the only common subgraph is a path or a subdivision of a star T' with $\Delta(T') < \Delta(T)$, contradicting the choice of T . \blacksquare

Theorem 4. *Let \mathcal{X} be a finite family of connected line graphs of triangle-free and cherry-free graphs such that $\text{Forb}(C, \mathcal{X})$ is stable. Then \mathcal{X} contains a path or a generalized net.*

Proof is immediate by Proposition 3. \blacksquare

In the case of infinite families of forbidden subgraphs an analogue of Corollary 4 is not true, as can be seen e.g. by considering the stable class $\mathcal{X}_k = \text{Forb}(C, \{C_i\}_{i=k}^{\infty})$ (for any fixed $k \geq 3$); \mathcal{X}_k is stable although neither of the forbidden subgraphs is a path or a generalized net. However, it is still possible to show that each family of forbidden subgraphs that yields a stable class contains a proper subfamily with the same property.

Proposition 5. *Let \mathcal{Y} be a split-closed family of connected graphs. Then \mathcal{Y} contains a proper subfamily $\mathcal{Y}' \subset \mathcal{Y}$, $\mathcal{Y}' \neq \mathcal{Y}$, such that $\mathcal{Y} \setminus \mathcal{Y}'$ is split-closed, unless $|\mathcal{Y}| = 1$ and $\mathcal{Y} = \{Y\}$, where Y is a path or a subdivision of the claw.*

Proof. First suppose that \mathcal{Y} contains no tree and set $g(\mathcal{Y}) = \min\{g(Y) \mid Y \in \mathcal{Y}\}$ and $\mathcal{Y}' = \{Y \in \mathcal{Y} \mid g(Y) = g(\mathcal{Y})\}$. Then $\mathcal{Y} \setminus \mathcal{Y}'$ is split-closed (since vertex splitting cannot reduce the length of a cycle).

Next suppose that \mathcal{Y} contains both a tree and some graph that is not a tree. If $\mathcal{Y}' \subset \mathcal{Y}$ is the subfamily of all graphs from \mathcal{Y} that are not trees, then $\mathcal{Y} \setminus \mathcal{Y}'$ is split-closed (since splitting a vertex of a tree cannot create a cycle).

Finally, suppose that all graphs in \mathcal{Y} are trees and, for a tree T , define the core $\text{co}(T)$ of T as the smallest subtree $T' \subset T$ such that $V_{\geq 3}(T - V_1(T)) \subset V(T')$, and set $\eta(T) = |\text{co}(T)|$ and $\eta(\mathcal{Y}) = \min\{\eta(T) \mid T \in \mathcal{Y}\}$. Let $\mathcal{Y}' = \{T \in \mathcal{Y} \mid \eta(T) = \eta(\mathcal{Y})\}$. If $\mathcal{Y}' \neq \mathcal{Y}$, then again $\mathcal{Y} \setminus \mathcal{Y}'$ is split-closed (since, by vertex splitting, $\eta(T)$ cannot decrease).

Hence it remains to consider the case when $\eta(T) = \eta(\mathcal{Y})$ for any $T \in \mathcal{Y}$. If $\eta(T) \geq 2$, or if $\eta(T) = 1$ and the vertex in $\text{co}(T)$ is of degree at least 4, then splitting of a vertex in $\text{co}(T)$ increases $\eta(T)$ and we are in the previous case. Thus, the only remaining possibility is that $\eta(T) \leq 1$ and $|V_{\geq 4}(T)| = 0$, ie., T is a path or a subdivision of the claw. ■

Theorem 6. *Let \mathcal{X} be a family of line graphs of connected triangle-free and cherry-free graphs such that $\text{Forb}(C, \mathcal{X})$ is stable. Then \mathcal{X} contains a subfamily $\mathcal{X}' \subset \mathcal{X}$, $\mathcal{X}' \neq \mathcal{X}$, such that $\text{Forb}(C, \mathcal{X} \setminus \mathcal{X}')$ is stable, unless $|\mathcal{X}| = 1$ and \mathcal{X} contains a path or a generalized net.*

Proof is immediate by Proposition 5. ■

5 Unstable classes and stabilizers

In this section we show a common background of the techniques that can be used to prove results similar to Theorem D and Corollary E, and to their analogue for the 2-factor closure proved in [13]. Throughout the section, \mathfrak{cl} will always denote a closure operation that turns a claw-free graph into a line graph (possibly of a multigraph) by a series of local completions at eligible vertices; the definition of eligibility can vary in specific cases. Thus, special cases of the closure operation \mathfrak{cl} are the closure introduced in [11], the cycle closure introduced in [4], the contraction closure introduced in [15], the 2-factor closure introduced in [16], the multigraph closure introduced in [17] or the strong M-closure introduced in [10].

Let \mathcal{B} be a family of connected line graphs of triangle-free and cherry-free graphs, \mathcal{G} a family of connected closed (under \mathfrak{cl}) claw-free graphs, and let $k \geq 2$ be an integer. Then a family \mathcal{S} of connected triangle-free and cherry-free graphs such that

- (i) \mathcal{S} is split-closed,
- (ii) every $S \in \mathcal{S}$ contains a subgraph isomorphic to $L^{-1}(B)$ for some $B \in \mathcal{B}$,
- (iii) every \mathfrak{cl} -closed k -connected $(C, L(\mathcal{S}))$ -free graph is (C, \mathcal{G}) -free,

is called a k -stabilizer for \mathcal{B} into \mathcal{G} under \mathfrak{cl} , or briefly a $(k, \mathcal{B}, \mathcal{G}, \mathfrak{cl})$ -stabilizer. In the special case when $\mathcal{B} = \{B\}$ and $\mathcal{G} = \{Q\}$ we simply say that \mathcal{S} is a (k, B, Q, \mathfrak{cl}) -stabilizer.

Remark. If $Q \in \mathcal{G}$ is not \mathfrak{cl} -closed, then clearly $\mathfrak{cl}(G)$ is Q -free for any claw-free graph G . Similarly, if for some $B \in \mathcal{B}$, $L^{-1}(B)$ is not triangle-free and cherry-free, then B is redundant since (ii) is an empty condition for B . In the definition of a stabilizer we suppose that all graphs in \mathcal{G} are closed and all graphs in $L^{-1}(\mathcal{B})$ are triangle-free and cherry-free in order to avoid such trivial cases.

The following theorem shows that if for some classes \mathcal{B}, \mathcal{G} there is a k -stabilizer, then the closures of all k -connected (C, \mathcal{B}) -free graphs are (C, \mathcal{G}) -free, i.e. $\text{cl}(\text{Forb}_k(C, \mathcal{B})) \subset \text{Forb}_k(C, \mathcal{G})$.

In typical applications (see [12], [13]) this technique was used to handle an unstable class $\text{Forb}(C, \mathcal{B})$ by inserting closures of graphs from $\text{Forb}(C, \mathcal{B})$ into a stable class $\text{Forb}(C, \mathcal{G})$ (which also motivates the name “stabilizer”), however, it turns out that the stability of $\text{Forb}(C, \mathcal{G})$ is not necessary for the inclusion.

Theorem 7. *Let $k \geq 1$, let \mathcal{B}, \mathcal{G} be classes of graphs such that there is a $(k, \mathcal{B}, \mathcal{G}, \text{cl})$ -stabilizer, and let G be a k -connected (C, \mathcal{B}) -free graph. Then $\text{cl}(G)$ is (C, \mathcal{G}) -free.*

Proof. Let \mathcal{S} be a $(k, \mathcal{B}, \mathcal{G}, \text{cl})$ -stabilizer, $k \geq 1$, and suppose that $\text{cl}(G)$ is not \mathcal{G} -free, i.e. $Q \stackrel{\text{IND}}{\subset} \text{cl}(G)$ for some $Q \in \mathcal{G}$. By (iii), $\text{cl}(G)$ is not $L(\mathcal{S})$ -free, i.e. there is an $S \in \mathcal{S}$ such that $L(S) \stackrel{\text{IND}}{\subset} \text{cl}(G)$. By (i), by Proposition 1 and by induction, $L(S') \stackrel{\text{IND}}{\subset} G$ for some $S' \in \mathcal{S}$. By (ii), G is not \mathcal{B} -free, a contradiction. ■

Examples. 1. Let $S_1, S_2, N_{1,1,2}$ be the graphs in Fig. 2 and set $S'_1 = L^{-1}(S_1)$, $S'_2 = L^{-1}(S_2)$, $S'_3 = L^{-1}(N_{1,1,2})$ and $\mathcal{S} = \{S'_1, S'_2, S'_3\}$. As already noted, \mathcal{S} is split-closed, and it is also not difficult to verify that a 2-connected closed graph G containing an induced $N_{1,1,1}$ must also contain some of S'_i , $i = 1, 2, 3$ (since then $L^{-1}(G)$ contains an $L^{-1}(N_{1,1,1})$ and the connectivity assumption implies the existence of some suitable additional paths). Equivalently, a closed (C, \mathcal{S}) -free graph must be $(C, N_{1,1,1})$ -free. Since each of S'_1, S'_2, S'_3 contains $L^{-1}(B_{1,2})$ as a subgraph, \mathcal{S} is a $(2, B_{1,2}, N_{1,1,1}, \text{cl})$ -stabilizer. This gives a much simpler proof of part (ii) of Theorem D and the subsequent simplification of the Bedrossian’s characterization (Corollary E).

2. A similar approach would give an alternative (and simpler) proof of the fact that closures of 2-connected (C, Z_3) -free graphs are $(C, N_{1,1,1})$ -free (originally in [12]), hence also immediately part (iii) of Theorem D. The proof is slightly more elaborate here since there are several finite exceptions and one infinite class of exceptions, nevertheless still much easier than in the original paper [12]. We leave details to the reader.

3. A similar approach was used in [13] to simplify the characterization of forbidden pairs for 2-factors. Having in mind the definition of a stabilizer, it is easy to observe that the family of “good suns”, as defined in [13], is a $(2, B_{1,3}, N_{1,1,2}, \text{cl}^{2f})$ -stabilizer (where cl^{2f} is the 2-factor closure introduced in [16]).

Remark. Although Theorem 4 shows that the only minimal families \mathcal{X} such that $\text{Forb}(C, \mathcal{X})$ is stable are the one-element ones containing a path or a generalized net, from Example 1 we see that an analogue for stabilizers is not true: the (finite) family $\{S_1, S_2, N_{1,1,2}\}$ is a $(2, B_{1,2}, N_{1,1,1}, \text{cl})$ -stabilizer, but it is easy to verify that none of its proper subfamilies has this property. However, a characterization of minimal (in some sense) stabilizers for given families \mathcal{B}, \mathcal{G} remains an open problem.

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