

# 2-factors with bounded number of components in claw-free graphs

Kenta Ozeki\*

*National Institute of Informatics,  
2-1-2 Hitotsubashi, Chiyoda-ku,  
Tokyo 101-8430, Japan*

and

*JST, ERATO, Kawarabayashi Large Graph Project  
e-mail: ozeki@nii.ac.jp*

Zdeněk Ryjáček<sup>†‡</sup>

*Department of Mathematics,  
University of West Bohemia,  
P.O. Box 314, 306 14 Pilsen, Czech Republic  
e-mail: ryjacek@kma.zcu.cz*

Kiyoshi Yoshimoto<sup>§</sup>

*Department of Mathematics  
College of Science and Technology, Nihon University  
Tokyo 101-8308, Japan  
e-mail: yosimoto@math.cst.nihon-u.ac.jp*

## Abstract

In this paper, we show that every 3-connected claw-free graph  $G$  has a 2-factor having at most  $\max\{\frac{2}{5}(\alpha + 1), 1\}$  cycles, where  $\alpha$  is the independence number of  $G$ . As a corollary of this result, we also prove that every 3-connected claw-free graph  $G$  has a 2-factor with at most  $(\frac{4|G|}{5(\delta+2)} + \frac{2}{5})$  cycles, where  $\delta$  is the minimum degree of  $G$ . This is an extension of a known result on the number of cycles of a 2-factor in 3-connected claw-free graphs.

Keywords: Claw-free graphs; 2-factors; Independence number

---

\*The authors' work was in part supported by JSPS KAKENHI Grant Number 25871053 and by Grant for Basic Science Research Projects from The Sumitomo Foundation.

<sup>†</sup>Institute for Theoretical Computer Science (ITI), Charles University, Prague, Czech Republic

<sup>‡</sup>Research supported by grant No. P202/12/G061 of the Czech Science Foundation.

<sup>§</sup>This author's work was supported by JSPS KAKENHI Grant Number 26400190.

# 1 Introduction

A well-known conjecture by Matthews and Sumner [17] states that every 4-connected claw-free graph is Hamiltonian. Recall that a graph is *claw-free* if it has no claw  $K_{1,3}$  as an induced subgraph. Thomassen [20] also posed the following conjecture: every 4-connected line graph is Hamiltonian. Note that Ryjáček [18] showed that these two conjectures are equivalent, using a closure technique. These two conjectures have attracted much attention during the last more than 25 years, but they are still open.

To attack these conjectures, some researchers have considered the Hamiltonicity of claw-free graphs with high connectivity conditions. In fact, Zhan [22], and independently Jackson [12] proved that Thomassen’s conjecture is true for 7-connected line graphs. Recently, Kaiser and Vrána [15] improved this result by showing that every 5-connected claw-free graph with minimum degree at least six is Hamiltonian. Like these, several researchers have attacked these conjectures in claw-free graphs with high connectivity. See for example [11, 23].

On the other hand, it is also natural to ask what happens when we consider claw-free graphs with low connectivity. Although it is known that there exist infinitely many 3-connected claw-free graphs (also line graphs) having no Hamiltonian cycles, we would like to find some “good” structures which have some properties close to Hamiltonian cycles in such graphs. The main target of this paper is a *2-factor* with a bounded number of components. (See the survey [7] for other “good” structures.)

Recall that a 2-factor of a graph is a spanning subgraph in which all vertices have degree two. A Hamiltonian cycle of a graph is actually a 2-factor with exactly one component. In this sense, the fewer components a 2-factor has, the closer to a Hamiltonian cycle it is. Choudum and Paulraj [4], and independently Egawa and Ota [5] proved that if the minimum degree of a claw-free graph  $G$  is at least four, then  $G$  has a 2-factor (without considering the number of components). Yoshimoto [21] showed that if  $G$  is a 2-connected claw-free graph with minimum degree at least three (specifically, if  $G$  is 3-connected), then  $G$  has a 2-factor.

Now we consider a 2-factor with bounded number of cycles in claw-free graphs. Faudree, Favaron, Flandrin, Li and Liu [6] showed that a claw-free graph with minimum degree  $\delta \geq 4$  has a 2-factor with at most  $\frac{6|G|}{\delta+2} - 1$  cycles. Gould and Jacobson [10] improved this result for a claw-free graph with large minimum degree; a claw-free graph with minimum degree  $\delta \geq (4|G|)^{\frac{2}{3}}$  has a 2-factor with at most  $\lceil \frac{|G|}{\delta} \rceil$  cycles. Recently, Broersma, Paulusma and Yoshimoto showed the following result.

**Theorem 1 (Broersma, Paulusma and Yoshimoto [1])** *Every claw-free graph  $G$  with minimum degree  $\delta \geq 4$  has a 2-factor with at most*

$$\max \left\{ \frac{|G| - 3}{\delta - 1}, 1 \right\} \text{ cycles.}$$

Also Yoshimoto [21] showed that the coefficient  $\frac{1}{\delta-1}$  of  $|G|$  is almost best possible.

Now we consider 2-connected or 3-connected claw-free graphs. Jackson and Yoshimoto [13] showed that every 2-connected claw-free graph  $G$  with minimum degree at least four has a 2-factor with at most  $\frac{|G|+1}{4}$  cycles, and moreover, with at most  $\frac{2|G|}{15}$  cycles if  $G$  is 3-connected. Čada, Chiba and Yoshimoto [2] proved that every 2-connected claw-free graph  $G$  with minimum degree  $\delta \geq 4$  has a 2-factor in which every cycle has the length at least  $\delta$ . This result implies the existence of a 2-factor with at most  $\frac{|G|}{\delta}$  cycles in a 2-connected claw-free graph  $G$ .

On the other hand, Kužel, Ozeki and Yoshimoto [16] focused on a relationship between a 2-factor and maximum independent sets in a graph, and showed the following:

**Theorem 2 (Kužel, Ozeki and Yoshimoto [16])** *For every maximum independent set  $S$  in a 2-connected claw-free graph  $G$  with minimum degree at least three,  $G$  has a 2-factor in which each cycle contains at least one vertex in  $S$ , and moreover, at least two vertices in  $S$  if  $G$  is 3-connected.*

As a direct corollary of Theorem 2, we obtain that every 3-connected claw-free graph  $G$  has a 2-factor with at most  $\alpha/2$  cycles, where  $\alpha$  is the independence number of  $G$ . Note that for every claw-free graph  $G$ , we have that  $\alpha \leq \frac{2|G|}{\delta+2}$ , where  $\alpha$  is the independence number and  $\delta$  is the minimum degree of  $G$ , respectively. (See for example, Fact 8 in [8].) Therefore, the result of Kužel et al. implies the following corollary.

**Theorem 3 (Kužel, Ozeki and Yoshimoto [16])** *Every 3-connected claw-free graph  $G$  with minimum degree  $\delta$  has a 2-factor with at most*

$$\max \left\{ \frac{|G|}{\delta+2}, 1 \right\} \text{ cycles.}$$

In this paper, we show the following result, which means that if we do not specify a maximum independent set, for 3-connected claw-free graphs, we can find a 2-factor with fewer cycles than the one obtained by Theorem 2.

**Theorem 4** *Every 3-connected claw-free graph with independence number  $\alpha$  has a 2-factor with at most*

$$\max \left\{ \frac{2}{5}(\alpha+1), 1 \right\} \text{ cycles.}$$

We do not know whether the coefficient  $\frac{2}{5}$  of  $\alpha$  in Theorem 4 is best possible or not. However, in Section 3, we show two examples to discuss sharpness of the result. By the same argument as above, Theorem 4 implies the following corollary.

**Corollary 5** *Every 3-connected claw-free graph  $G$  with minimum degree  $\delta$  has a 2-factor with at most*

$$\left( \frac{4|G|}{5(\delta+2)} + \frac{2}{5} \right) \text{ cycles.}$$

In Corollary 5, we decrease the coefficient of  $|G|$  in Theorem 3. This is the first result that guarantees, in a 3-connected claw-free graph, the existence of a 2-factor having number of cycles with coefficient of  $|G|/\delta$  less than 1.

In the next section, we give two statements (Theorems 6 and 7), that are equivalent to Theorem 4. After discussing sharpness of Theorem 4 in Section 3, we show some lemmas in Sections 4 and 5. In Section 6, we prove Theorem 7.

## 2 Preliminaries

For a graph  $G$  and for  $S \subset V(G)$ ,  $G[S]$  denotes the subgraph of  $G$  induced by the set  $S$ . We denote by  $N_G(x)$  the neighborhood of a vertex  $x$  in a graph  $G$ .

For the proof of Theorem 4, we use the *closure* of a claw-free graph which was introduced by Ryjáček [18] as follows. For each vertex  $x$  of a claw-free graph  $G$ ,  $N_G(x)$  induces a subgraph  $G[N_G(x)]$  with at most two components, and if  $G[N_G(x)]$  has two components, both of them must be cliques. In the case where  $G[N_G(x)]$  is connected and non-complete, we add edges joining all pairs of nonadjacent vertices in  $N_G(x)$ . The closure  $\text{cl}(G)$  of  $G$  is the (unique) graph obtained by recursively repeating this operation, as long as this is possible. Ryjáček, Saito and Schelp [19] proved that a claw-free graph  $G$  has a 2-factor with at most  $c$  components if and only if  $\text{cl}(G)$  has a 2-factor with at most  $c$  components. This implies that the following statement is equivalent to Theorem 4.

**Theorem 6** *For every 3-connected claw-free graph  $G$  with independence number  $\alpha$ ,  $\text{cl}(G)$  has a 2-factor with at most  $\max\{\frac{2}{3}(\alpha + 1), 1\}$  cycles.*

Ryjáček [18] proved that for every claw-free graph  $G$ , there exists a triangle-free simple (i.e. with no parallel edges) graph  $H$  such that  $L(H) = \text{cl}(G)$ . An *even graph* is a graph in which all vertices have even degree, and a *circuit* is a connected even graph. Let  $H$  be a graph. A set  $\mathcal{D}$  of circuits and stars with at least three edges in  $H$  is called a *D-system of  $H$* , if every edge of  $H$  is contained in a member of  $\mathcal{D}$  or incident with a vertex in a circuit in  $\mathcal{D}$ . For a *D-system  $\mathcal{D}$  of  $H$* , let  $|\mathcal{D}|$  be the number of circuits and stars in  $\mathcal{D}$ . Also a *D-system  $\mathcal{D}$  of  $H$*  is called a *strong D-system of  $H$*  if  $\mathcal{D}$  contains no star and every vertex of degree at least three in  $H$  is contained in some circuit in  $\mathcal{D}$ . Gould and Hynds [9] proved that the line graph  $L(H)$  of a graph  $H$  has a 2-factor with  $c$  components if and only if there is a *D-system  $\mathcal{D}$  of  $H$*  with  $|\mathcal{D}| = c$  in  $H$ . An edge set  $E_0$  of  $H$  is called an *essential edge-cut* if  $H - E_0$  contains at least two components having an edge. A graph  $H$  is *essentially  $k$ -edge connected* if there exists no essential edge-cut with at most  $k - 1$  edges. Clearly  $L(H)$  is  $k$ -connected if and only if  $H$  is essentially  $k$ -edge-connected. Let  $\alpha'(H)$  be the number of edges of a maximum matching of  $H$ . Note that when  $L(H) = G$ , then  $\alpha(G) = \alpha'(H)$ . Then the following is also equivalent to Theorems 4 and 6.

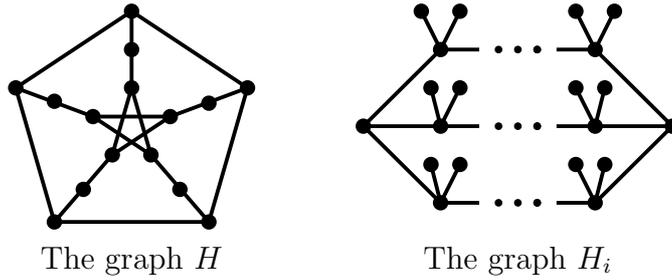


Figure 1: The graphs  $H$  and  $H_i$ .

**Theorem 7** *Let  $G$  be an essentially 3-edge connected graph. Then  $G$  has a  $D$ -system  $\mathcal{D}$  with  $|\mathcal{D}| \leq \max\{\frac{2}{5}(\alpha'(G) + 1), 1\}$ .*

Note that the statement of Theorem 7 remains equivalent to Theorems 4 and 6 even if restricted to triangle-free simple graphs; however, its present form will be more convenient for our proof.

An edge  $e = uv$  of a graph  $G$  is said to be *pendant* if the degree of  $u$  or  $v$  is one in  $G$ . For an integer  $l \geq 2$ , the cycle of length  $l$  is denoted by  $C_l$ . For an integer  $g \geq 2$ ,  $K_{2,2g}$  denotes the complete bipartite graph such that one partite set consists of two vertices and the other consists of  $2g$  vertices. For a subgraph  $H$  of a graph  $G$  and for a set  $E_1$  of edges in  $G - E(H)$ , we define  $H + E_1$  as the graph induced by edges  $E(H) \cup E_1$ .

### 3 Sharpness of Theorem 4

In this section, we discuss how far Theorem 4 is from being sharp. First, we consider the upper bound on the number of components of a 2-factor. Although we do not know whether the coefficient  $\frac{2}{5}$  of  $\alpha$  (or  $\alpha'$  in Theorem 7) is best possible or not, the following graph shows that it cannot be less than  $\frac{2}{7}$ . Let  $H_0$  be the Petersen graph, let  $M_0$  be a maximum matching of  $H_0$  and let  $H$  be the graph obtained from  $H_0$  by subdividing all edge in  $M_0$  once (see the left side of Figure 1). Suppose that  $H$  has a  $D$ -system  $\mathcal{D}$  with  $|\mathcal{D}| = 1$ , say,  $\{D\} = \mathcal{D}$ . Since  $D$  has to pass through all vertices of  $H_0$  (because otherwise  $D$  cannot dominate a subdivided edge in  $H$  incident with a vertex not passed by  $D$ ),  $D$  corresponds to a Hamiltonian cycle of the Petersen graph  $H_0$ , a contradiction. Thus, every  $D$ -system of  $H$  has at least two members. Since  $\alpha(L(H)) = \alpha'(H) = 7$ , the coefficient of  $\alpha'$  in Theorem 7 has to be at least  $\frac{2}{7}$ . Considering the graph  $L(H)$ , we can also show that the coefficient of  $\alpha$  in Theorem 4 has to be at least  $\frac{2}{7}$ .

Next we consider the 3-connectedness (or essential 3-edge connectedness) assumption in Theorem 4 (Theorem 7, respectively). Unfortunately, we do not know whether the 2-connectedness (or essential 2-edge connectedness) might be sufficient for the result, but we give examples which show that we cannot decrease the coefficient to

less than  $\frac{1}{3}$  if we only assume 2-connectedness (or essential 2-edge connectedness, respectively). Let  $H'_i$  be obtained from the graph with two vertices and three internally disjoint paths each of which contains  $i$  internal vertices. We add two pendant edges to each internal vertex of  $H'_i$  and obtain the graph  $H_i$  (see the right side of Figure 1). Note that  $\alpha'(H_i) = 3i$ . Since every circuit of  $H_i$  has to miss at least one of the three paths of  $H_i$ , each  $D$ -system  $\mathcal{D}_i$  of  $H_i$  has at least  $i$  stars, so it has at least  $i + 1$  members. This implies that

$$\lim_{i \rightarrow \infty} \frac{|\mathcal{D}_i|}{\alpha'(H_i)} \geq \lim_{i \rightarrow \infty} \frac{i + 1}{3i} = \frac{1}{3}.$$

## 4 Contractions and reconstructions

### 4.1 Contractions used in this paper

In this paper, in order to make a given graph smaller, we consider the following six types of contractions. Also, we use the reverse operation of those, called *reconstructions*. Let  $G$  be a graph. (Possibly,  $G$  may have multiple edges.)

#### A suppressing:

Let  $x$  be a vertex of degree two and let  $e$  be an edge incident with  $x$ . A *suppressing (of  $x$ )* is a contraction of the edge  $e$  to one vertex and removing the created loop.

#### A $C_2$ -contraction, a $C_3$ -contraction and a primary $K_{2,2g}$ -contraction:

Let  $C$  be a cycle of length two in  $G$ . A  $C_2$ -contraction (at  $C$ ) consists of the following three operations, executed in order:

- contracting  $C$  to one vertex,
- removing all created loops,
- adding a new pendant edge to the contracted vertex.

When  $C$  is a cycle of length three in  $G$  or a subgraph isomorphic to  $K_{2,2g}$  with an integer  $g \geq 2$ , we define similarly a  $C_3$ -contraction (at  $C$ ) or a *primary  $K_{2,2g}$ -contraction*, respectively.

#### A secondary $K_{2,2g}$ -contraction:

Let  $C$  be a subgraph of  $G$  isomorphic to  $K_{2,2g}$  for some  $g \geq 2$ . Let  $x_1, x_2$  be the two vertices of the smaller partite set of  $C$ , and let  $Y$  be the other partite set. For  $Y_1 \subset Y$  with  $Y_1 \neq \emptyset$  and  $Y_1 \neq Y$ , a *secondary  $K_{2,2g}$ -contraction at  $C$  with respect to  $Y_1$*  consists of the following five operations, executed in order:

- identifying all vertices in  $Y_1$  to one vertex, say  $y_1$ ,

- identifying all vertices in  $Y \setminus Y_1$  to one vertex, say  $y_2$ ,
- replacing multiple edges between  $x_i$  and  $y_j$  with a single edge for  $i, j = 1, 2$ ,
- removing all loops,
- removing all pendant edges incident with  $x_1$  or  $x_2$ .

Note that although the original graph is simple, the graph obtained by a secondary  $K_{2,2g}$ -contraction might have multiple edges between  $y_1$  (or  $y_2$ ) and some vertex  $z$  with  $z \neq x_1, x_2$ .

#### A $C_5$ -contraction:

Let  $C$  be a cycle of length five. A  $C_5$ -contraction (at  $C$ ) consists of the following two operations, executed in order:

- contracting  $C$  to one vertex,
- removing all created loops.

## 4.2 3-edge connectedness

In this subsection, we consider 3-edge connectedness of a graph obtained by the contractions defined in Section 4.1. By the definition, the following is an easy fact.

**Fact 8** *Let  $G$  be an essentially 3-edge connected graph, and let  $G'$  be a graph obtained from  $G$  by a suppressing, a  $C_2$ -contraction, a  $C_3$ -contraction, a primary  $K_{2,2g}$ -contraction, or by a  $C_5$ -contraction. Then  $G'$  is essentially 3-edge connected.*

On the other hand, for a secondary  $K_{2,2g}$ -contraction, we show the following useful lemma. For a subgraph  $C$  of  $G$  isomorphic to  $K_{2,2g}$  with  $g \geq 2$ ,  $C$  is called *good* if all but at most two vertices in  $Y$  have degree two in  $G$ , where  $Y$  is the larger partite set of  $C$ , and  $C$  is *bad* if  $C$  is not good.

**Lemma 9** *Let  $G$  be an essentially 3-edge connected graph and let  $C$  be a subgraph of  $G$  isomorphic to  $K_{2,2g}$  with  $g \geq 2$ . Let  $x_1, x_2$  be the two vertices of the smaller partite set of  $C$  and let  $Y$  be the other partite set. Suppose that  $C$  is bad. Then one of the following holds:*

- (i) *for some  $i = 1, 2$ , all edges of  $C$  incident with  $x_i$  form an essential edge-cut of  $G$ ,*
- (ii) *there exists a subset  $Y_1 \subset Y$  with  $Y_1 \neq \emptyset$  and  $Y_1 \neq Y$  such that the graph obtained by a secondary  $K_{2,2g}$ -contraction at  $C$  with respect to  $Y_1$  is also essentially 3-edge connected.*

**Proof of Lemma 9.** Let  $C, x_1, x_2, Y$  be as in the assumptions of Lemma 9 and suppose that  $C$  is bad. We first claim that there exists a path  $P$  in  $G - E(C)$  connecting some two vertices in  $Y$ .

Since  $C$  is not good, there exist three vertices  $y^1, y^2, y^3 \in Y$  such that  $d_G(y^i) \geq 3$  for  $i = 1, 2, 3$ . Since  $G$  is essentially 3-edge connected, there exists a path  $P_1$  in  $G - \{x_1y^1, x_2y^1\}$  from  $y^1$  to  $V(C) \setminus \{y^1\}$ . Note that  $P_1$  is a path in  $G - E(C)$ . If  $P_1$  reaches  $y$  for some  $y \in Y \setminus \{y^1\}$ , then  $P_1$  is the desired path. Thus we may assume that  $P_1$  reaches  $x_1$  or  $x_2$ . Similarly, we can take two paths  $P_2$  and  $P_3$  in  $G - E(C)$  from  $y^2$  and  $y^3$ , respectively, to  $x_1$  or  $x_2$ . Since at least two of the paths  $P_1, P_2, P_3$  have the same end vertex in  $\{x_1, x_2\}$ , connecting them, we can find a path  $P$  in  $G - E(C)$  between two vertices in  $Y$ .

Let  $y^1$  and  $y^2$  be the end vertices of the path  $P$  in  $G - E(C)$ . Let  $Y_1 := \{y^1\}$  and consider the graph  $G'$  obtained from  $G$  by a secondary  $K_{2,2g}$ -contraction at  $C$  with respect to  $Y_1$ . Let  $y_1, y_2$  be the vertices of  $G'$  obtained from  $Y_1$  and  $Y \setminus Y_1$ , respectively.

Suppose that (ii) does not hold, that is, there exists an essential edge-cut  $E_1 \subset E(G')$  of  $G'$  with  $|E_1| \leq 2$ . If  $|E_1 \cap \{x_1y_1, x_2y_1, x_1y_2, x_2y_2\}| = 0$ , then  $E_1$  is also an essential edge-cut of  $G$ , a contradiction. If  $|E_1 \cap \{x_1y_1, x_2y_1, x_1y_2, x_2y_2\}| = 1$ , say  $x_1y_1 \in E_1$ , then  $x_1$  and  $y_1$  are contained in the same component of  $G' - E_1$  because  $G' - E_1$  has the path  $x_1y_2x_2y_1$ , a contradiction again. Therefore  $|E_1| = 2$  and  $E_1 \subset \{x_1y_1, x_2y_1, x_1y_2, x_2y_2\}$ .

Since  $P$  connects  $y^1$  and  $y^2$  in  $G - E(C)$ , it also connects  $y_1$  and  $y_2$  in  $G' - E_1$ . This implies that  $y_1$  and  $y_2$  are contained in the same component of  $G' - E_1$ , and hence  $E_1 = \{x_iy_1, x_iy_2\}$  for some  $i = 1, 2$ . Since  $E_1$  is an essential edge-cut of  $G'$ , there exists a component  $H_1$  of  $G' - E_1$  with  $x_i \in V(H_1)$ ,  $y_1, y_2 \notin V(H_1)$  and  $|H_1| \geq 2$ . Since we removed all pendant edges incident with  $x_i$ , the edges of  $C$  incident with  $x_i$  correspond to edges in  $E_1$ , and hence they form an essential edge-cut of  $G$ . Thus, (i) holds.  $\square$

### 4.3 Reconstructions of a $C_2$ - or $C_3$ -contraction

In this subsection, we deal with reconstructions of a  $C_2$ - or  $C_3$ -contraction. The first statement can be found in several papers, for example [3], and the second one can be easily shown. Hence we omit the proof.

**Lemma 10** *Let  $G$  be a graph and let  $C$  be a cycle of length two or three in  $G$ . Let  $G'$  be the graph obtained from  $G$  by a  $C_2$ - or  $C_3$ -contraction at  $C$ . Then:*

- (i) *If  $G'$  has a  $D$ -system  $\mathcal{D}'$ , then  $G$  also has a  $D$ -system  $\mathcal{D}$  with  $|\mathcal{D}| \leq |\mathcal{D}'|$ . In particular, if  $\mathcal{D}'$  is strong, then  $\mathcal{D}$  is also strong.*

- (ii) For any matching  $M'$  in  $G'$ , there exists a matching  $M$  in  $G$  with  $|M| \geq |M'|$ . In particular,  $\alpha'(G) \geq \alpha'(G')$ .

#### 4.4 Reconstructions of $K_{2,2g}$ -contractions

In this subsection, we deal with reconstructions of a primary or secondary  $K_{2,2g}$ -contraction. Indeed, we show the following lemma.

**Lemma 11** *Let  $G$  be a triangle-free graph and let  $C$  be a bad  $K_{2,2g}$  for some  $g \geq 2$  in  $G$ . Then all of the following hold:*

- (i) *Suppose that  $C$  satisfies condition (i) in Lemma 9. Let  $G'$  be the graph obtained by a primary  $K_{2,2g}$ -contraction at  $C$ . If  $G'$  has a  $D$ -system  $\mathcal{D}'$ , then  $G$  also has a  $D$ -system  $\mathcal{D}$  with  $|\mathcal{D}| \leq |\mathcal{D}'|$ .*
- (ii) *Let  $G'$  be the graph obtained by a secondary  $K_{2,2g}$ -contraction at  $C$ . If  $G'$  has a  $D$ -system  $\mathcal{D}'$ , then  $G$  also has a  $D$ -system  $\mathcal{D}$  with  $|\mathcal{D}| \leq |\mathcal{D}'|$ .*
- (iii) *Let  $G'$  be the graph obtained by a primary or secondary  $K_{2,2g}$ -contraction at  $C$ . Then  $\alpha'(G) \geq \alpha'(G')$ .*

**Proof of Lemma 11.**

Since (iii) is obvious, we show only (i) and (ii) at the same time. Let  $G'$  be the graph obtained by a primary or secondary  $K_{2,2g}$ -contraction at  $C$  as in the statement (i) or (ii). Suppose that  $G'$  has a  $D$ -system  $\mathcal{D}'$ .

Let  $x_1, x_2$  be the vertices of the smaller partite set of  $C$  and let  $Y$  be the other partite set. Since  $G$  is triangle-free,  $Y$  is an independent set. Let  $H'$  be the subgraph of  $G'$  such that  $V(H')$  is the set of vertices which are contained in some circuit in  $\mathcal{D}'$  or are centers of some star in  $\mathcal{D}'$ , and  $E(H')$  is the set of edges in some circuit of  $\mathcal{D}'$ . Note that  $H'$  is an even subgraph of  $G'$ , and the number of components of  $H'$ , denoted by  $\omega(H')$ , is at most  $|\mathcal{D}'|$ . Notice also that, when (i) occurs, then every edge in  $H'$  is also an edge in  $G$ , and when (ii) occurs, then every edge in  $H'$  except for  $x_1y_1, x_1y_2, x_2y_1$  and  $x_2y_2$  is also an edge in  $G$ , where  $y_1$  and  $y_2$  are defined in the operations in a secondary  $K_{2,2g}$ -contraction. (Recall that we did not replace multiple edges of  $G'$  with a single edge, except for the third operation of a secondary  $K_{2,2g}$ -reduction.) Hence we can regard  $E(H')$  for (i) and  $E(H') \setminus \{x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$  for (ii) as a subset of edges in  $G$ . Let further  $\tilde{H}$  be the graph with  $V(\tilde{H}) = (V(H') - \{v_C\}) \cup Y \cup \{x_1, x_2\}$  and  $E(\tilde{H}) = E(H')$  for (i), where  $v_C$  is the vertex obtained by contracting  $C$ ; or with  $V(\tilde{H}) = (V(H') - \{y_1, y_2\}) \cup Y$  and  $E(\tilde{H}) = E(H') \setminus \{x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$  for (ii). Since all edges in  $\tilde{H}$  also appear in  $G$  in either case, we can regard  $\tilde{H}$  as a subgraph of  $G$ . Note that  $V(\tilde{H})$  dominates all edges in  $G$ , and all vertices except for (some of the) vertices in  $\{x_1, x_2\} \cup Y$  have even degrees in  $\tilde{H}$  (possibly the degree might be zero).

In the rest of the proof, we will construct an even subgraph  $H$  of  $G$  by adding some edges of  $C$  into  $\tilde{H}$  in such a way that the number of components of  $H$  (denoted  $\omega(H)$ ) does not exceed  $\omega(H')$ , and  $x_1$  and  $x_2$  are contained in the same component of  $H$ . Then, since all edges in  $G$ , which do not appear in  $G'$ , are incident with  $x_1$  or  $x_2$ ,  $V(H)$  dominates all edges in  $G$  and, since each isolated vertex  $v$  of  $H$  is also isolated in  $H'$ ,  $v$  is a center of some star  $D_v$  in  $\mathcal{D}'$ . Therefore the system

$$\mathcal{D} = \{D : D \text{ is a component of } H\} \cup \{D_v : v \text{ is isolated in } H\}$$

is a  $D$ -system of  $G$  such that  $|\mathcal{D}| = \omega(H) \leq \omega(H') \leq |\mathcal{D}'|$  and we are done.

Let  $Y_{odd}$  (or  $Y_{even}$ ) be the set of vertices  $y$  in  $Y$  having odd (or even, respectively) degree in  $\tilde{H}$ . Note that  $|Y_{odd}| + |Y_{even}| = 2g$ , that is,  $|Y_{odd}| + |Y_{even}|$  is even. Notice also that  $d_{\tilde{H}}(x_1) + d_{\tilde{H}}(x_2) + |Y_{odd}|$  is even since  $H'$  is an even subgraph of  $G'$ . We consider the following three cases, depending on the parities of the degrees of  $x_1$  and  $x_2$  in  $\tilde{H}$ .

**Case 1.** Both  $x_1$  and  $x_2$  have even degree in  $\tilde{H}$ .

In this case,  $|Y_{odd}|$  is even, and hence  $|Y_{even}|$  is also even.

Suppose first that  $Y_{even} \neq \emptyset$ . Then let

$$H := \tilde{H} + \{x_1y : y \in Y\} + \{x_2y : y \in Y_{even}\}.$$

By the choice, every vertex of  $G$  has even degree in  $H$ . Since  $Y_{even} \neq \emptyset$ , all vertices in  $\{x_1, x_2\} \cup Y$  are contained in the same component in  $H$ , and hence we have  $\omega(H) \leq \omega(H')$ . So,  $H$  has the desired properties.

Thus, we may assume that  $Y_{even} = \emptyset$ , that is,  $Y = Y_{odd}$ . Since  $H'$  is an even subgraph of  $G'$ , there exists a path  $P$  in  $\tilde{H}$  such that either (a)  $P$  connects a vertex in  $Y_1$ , say  $y^1$ , and  $x_i$  for some  $i = 1, 2$ , say  $i = 2$ , or (b)  $P$  connects two vertices in  $Y_1$ , say  $y^1$  and  $y^2$ . Then we divide  $Y_{odd}$  into two sets  $Y_{odd}^1$  and  $Y_{odd}^2$  so that both  $Y_{odd}^1$  and  $Y_{odd}^2$  has even number of vertices,  $y^1 \in Y_{odd}^1$ , and  $y^2 \in Y_{odd}^2$  if (b) occurs. Then let

$$H := \tilde{H} + \{x_1y : y \in Y_{odd}^1\} + \{x_2y : y \in Y_{odd}^2\}.$$

Also every vertex of  $G$  has an even degree in  $H$ . Because of the path  $P$  in  $\tilde{H}$ , all vertices in  $\{x_1, x_2\} \cup Y$  are contained in the same circuit in  $H$ . Hence  $H$  is a desired even subgraph.  $\square$

**Case 2.** One of  $x_1$  and  $x_2$  has an even degree and the other has an odd degree in  $\tilde{H}$ .

By symmetry, we may assume that  $x_1$  has an even degree and  $x_2$  has an odd degree in  $\tilde{H}$ . Note that  $|Y_{odd}|$  is odd, and hence  $|Y_{even}|$  is also odd. Let

$$H := \tilde{H} + \{x_1y : y \in Y\} + \{x_2y : y \in Y_{even}\}.$$

Then every vertex of  $G$  has an even degree in  $H$  and  $\omega(H) \leq \omega(H')$ , and hence  $H$  is a desired even subgraph.  $\square$

**Case 3.** Both  $x_1$  and  $x_2$  have an odd degree in  $\tilde{H}$ .

For (i), we supposed that  $C$  satisfies condition (i) in Lemma 9, that is, for some  $i = 1, 2$ , say  $i = 1$ , all edges of  $C$  incident with  $x_1$  form an essential edge-cut of  $G$ . Then by the construction, one component, say  $R$ , of  $\tilde{H}$  contains  $x_1$  but does not contain any vertices in  $Y \cup \{x_2\}$ . Then  $x_1$  is the unique vertex of odd degree in  $R$ , a contradiction. Thus, in this case, we need to consider only (ii), and we performed a secondary  $K_{2,2g}$ -contraction at  $C$ .

**Case 3.1.**  $Y_{\text{even}} = \emptyset$ .

Since  $y_1$  has an even degree in  $H'$ , as in Case 1, there exists a path in  $\tilde{H}$  connecting two vertices in  $Y_1$ , say,  $y^1$  and  $y^2$ . Then we divide  $Y_{\text{odd}}$  into two sets  $Y_{\text{odd}}^1$  and  $Y_{\text{odd}}^2$  such that both  $Y_{\text{odd}}^1$  and  $Y_{\text{odd}}^2$  have odd number of vertices and  $y^i \in Y_{\text{odd}}^i$  for  $i = 1, 2$ . Then let

$$H := \tilde{H} + \{x_1y : y \in Y_{\text{odd}}^1\} + \{x_2y : y \in Y_{\text{odd}}^2\}.$$

Then  $H$  is a desired even subgraph.  $\square$

**Case 3.2.**  $Y_{\text{even}} \neq \emptyset$  and  $Y_{\text{odd}} \neq \emptyset$ .

Since  $|Y_{\text{odd}}|$  is even, we can divide  $Y_{\text{odd}}$  into two sets  $Y_{\text{odd}}^1$  and  $Y_{\text{odd}}^2$  such that both  $Y_{\text{odd}}^1$  and  $Y_{\text{odd}}^2$  have odd number of vertices. Let

$$H := \tilde{H} + \{x_1y : y \in Y_{\text{odd}}^1 \cup Y_{\text{even}}\} + \{x_2y : y \in Y_{\text{odd}}^2 \cup Y_{\text{even}}\}.$$

Then  $H$  is a desired even subgraph.  $\square$

**Case 3.3.**  $Y_{\text{odd}} = \emptyset$ .

In this case, since  $x_1$  has odd degree in  $\tilde{H}$  and we performed a secondary  $K_{2,2g}$ -contraction at  $C$ , exactly one of the edges  $x_1y_1$  and  $x_1y_2$  is used in  $H'$ . By symmetry, we may assume that  $x_1y_1$  is used in  $H'$ . Similarly, exactly one of the edges  $x_2y_1$  and  $x_2y_2$  is used in  $H'$ . Let  $D_1$  be the circuit in  $H'$  using the edge  $x_1y_1$ . If  $y_2$  is neither contained in any circuit of  $\mathcal{D}'$  nor a center of any star in  $\mathcal{D}'$ , then every edge of  $G'$  incident with  $y_2$  are dominated by some vertex in  $H'$ , and hence for some  $y \in Y \setminus Y_1$ ,

$$H := \tilde{H} + \{x_iy' : i = 1, 2, y' \in Y \setminus \{y\}\}$$

is the desired even subgraph of  $G$ . So we may assume that  $y_2$  is passed by some circuit  $D_2 \in \mathcal{D}'$  or is a center of some star  $D_2 \in \mathcal{D}'$ .

When  $D_1 = D_2$ , there exists a path  $P$  in  $\tilde{H}$  connecting  $y_2$  and a vertex in  $\{x_1, x_2, y_1\}$ . Let  $y \in Y \setminus Y_1$  such that  $P$  starts from  $y$  in  $\tilde{H}$ . When  $D_1 \neq D_2$ , then we let  $y \in Y - Y_1$  be an arbitrary vertex. Let

$$H := \tilde{H} + \{x_iy' : i = 1, 2, y' \in Y \setminus \{y\}\}.$$

Then  $H$  is an even subgraph of  $G$ . If  $D_1 = D_2$ , then since  $H$  also has a path  $P$ , all vertices in  $\{x_1, x_2\} \cup Y$  are contained in the same component of  $H$ . Thus, we have  $\omega(H) \leq \omega(H')$ . On the other hand, if  $D_1 \neq D_2$ , then  $\{x_1, x_2\} \cup Y$  are contained in at most two components of  $H$ . Since  $x_1, x_2, y_1, y_2$  are contained in the two components  $D_1$  and  $D_2$  of  $H'$ , we also have that  $\omega(H) \leq \omega(H')$ . In either case,  $H$  is the desired even subgraph. This completes the proof of Lemma 11.  $\square$

## 5 Lemmas

We use the following theorem in the proof of Theorem 7. Recall that a graph is called *even* if all its vertices have even degree.

**Theorem 12 (Jackson and Yoshimoto [13])** *Let  $G$  be a 3-edge connected graph of order  $n$ . Then  $G$  has a spanning even subgraph in which every component has at least  $\min\{5, n\}$  vertices.*

In the proof of Theorem 7, we will also often use the following observation.

**Fact 13** *Let  $C \simeq C_5$  be a subgraph of a graph  $G$ . Then for any edge  $uv$  incident with a vertex of  $C$ , say  $u \in V(C)$  and  $v \notin V(C)$ , there exists a matching in  $G[V(C) \cup \{v\}]$  with three edges.*

The next lemma concerns the existence of a matching with two or three edges in a circuit. A graph obtained from a star by replacing all edges with multiple edges is called a *flower*.

**Lemma 14** *Let  $D$  be a circuit of order at least four ( $D$  might possibly have multiple edges). Then:*

- (i)  $D$  has a matching with two edges unless  $D$  is a flower,
- (ii) If  $D$  has at least five vertices and contains no cycle of length two or three, then
  - ( $\alpha$ ) for all  $u \in D$ ,  $D - u$  has a matching with two edges, unless  $D \simeq K_{2,2g}$  for some  $g \geq 2$ ,
  - ( $\beta$ )  $D$  has a matching with three edges, unless  $D \simeq C_5$  or  $D \simeq K_{2,2g}$  for some  $g \geq 2$ .

**Proof.** If  $D$  contains a cycle of length at least six, then we can easily find a matching with three edges in  $D$ , and a matching with two edges in  $D - u$  for each  $u \in V(D)$ . Then we may assume that  $D$  contains no cycle of length at least six. If  $D$  has a cycle of length five,  $D$  has a matching with two edges. Moreover, if  $D$  contains no cycle

of length two or three, then by Fact 13,  $D$  has a matching with at least three edges, or  $D \simeq C_5$ . So all the statements in (i) and (ii) hold.

Thus, we may assume that  $D$  has no cycle of length at least five. Suppose next that  $D$  has a cycle  $C$  of length four, say,  $C = x_1x_2x_3x_4$ . Clearly,  $D$  has a matching with two edges, so the statement (i) holds. Suppose that  $D$  contains no cycle of length two or three. If there exists an edge in  $D - \{x_1, x_2, x_3, x_4\}$ , then we can find a matching with three edges, and hence (ii- $\alpha$ ) and (ii- $\beta$ ) hold. So we may assume that the cycle  $x_1x_2x_3x_4$  dominates all edges in  $D$ . On the other hand, if some vertex  $y$  in  $D - \{x_1, x_2, x_3, x_4\}$  has consecutive neighbors in  $C$ , we can find a cycle of length five, a contradiction. This implies that for any vertex  $y$  in  $D - \{x_1, x_2, x_3, x_4\}$ ,  $N_D(y) = \{x_1, x_3\}$  or  $N_D(y) = \{x_2, x_4\}$ . If there exist two vertices  $y_1, y_2$  in  $D - \{x_1, x_2, x_3, x_4\}$  with  $N_D(y_1) = \{x_1, x_3\}$  and  $N_D(y_2) = \{x_2, x_4\}$ , then  $y_1x_1x_2y_2x_4x_3y_1$  is a cycle of  $D$ , a contradiction. Thus, we may assume that  $N_D(y) = \{x_1, x_3\}$  for any vertex  $y$  in  $D - \{x_1, x_2, x_3, x_4\}$ , and hence  $N_D(x_1) = V(D) \setminus \{x_1, x_3\}$ . Since  $D$  is a circuit,  $|V(D) \setminus \{x_1, x_3\}|$  is even. Thus,  $D \simeq K_{2,2g}$ , where  $2g = |V(D) \setminus \{x_1, x_3\}|$ .

Next, we assume that  $D$  has no cycle of length at least four. Then  $D$  contains a cycle of length two or three, and hence it is enough to show only the statement (i). Now suppose that  $D$  has no matching with two edges. If  $D$  has a cycle  $C$  of length three, say,  $C = x_1x_2x_3$ , then there exists an edge  $yx_i$  ( $y \neq x_1, x_2, x_3$ ) in  $D$  for some  $i$ , say  $i = 1$ , since  $D$  is connected and  $D$  has at least four vertices. Then  $yx_1$  and  $x_2x_3$  form a matching with two edges, a contradiction. So we may assume that  $D$  has no cycle of length at least three, that is,  $D$  consists only of cycles isomorphic to  $C_2$ . If there exist two vertex disjoint cycles isomorphic to  $C_2$  in  $D$ , then taking one edge from each cycle, we obtain a matching with two edges. Thus, any two cycles share a vertex. This implies that  $D$  is a flower. This completes the proof of Lemma 14.  $\square$

## 6 Proof of Theorem 7

We use induction on  $|G|$ . When  $|G| \leq 5$ , we can easily find a desired  $D$ -system. Thus we may assume that  $|G| \geq 6$  and for all graphs with at most  $|G| - 1$  vertices the statement is true.

We divide the proof into five steps. In the first step (Subsection 6.1) we consider some contractions defined in Section 4.1 as a preliminary for  $C_5$ -contractions in the second step (Subsection 6.2), where, in the contracted graph, we also construct a strong  $D$ -system with ‘‘bounded number’’ of components. In the remaining three steps (Subsections 6.3 to 6.5), we will reconstruct all contracted  $C_5$ s one by one. During the reconstruction, in Subsection 6.4 we construct a ‘‘sufficiently large’’ matching, which will be in Subsection 6.5 completed a matching satisfying the statement of Theorem 7.

## 6.1 $C_2$ - or $C_3$ -contractions and $K_{2,2g}$ -contractions

In this subsection, we show the following two claims. Note that the first one is obvious by Lemmas 10 (i) and (ii).

**Claim 1**  $G$  has no cycle isomorphic to  $C_2$  or  $C_3$ , that is,  $G$  is simple and triangle-free.

**Claim 2**  $G$  has no bad  $K_{2,2g}$  for any  $g \geq 2$ .

**Proof.** Suppose not and let  $C$  be a bad  $K_{2,2g}$  for some  $g \geq 2$ . Let  $x_1, x_2$  be the vertices of the smaller partite set of  $C$  and let  $Y$  be the other partite set. By Lemma 9,  $C$  satisfies one of properties (i) and (ii) in Lemma 9. When (i) holds in Lemma 9, let  $G'$  be the graph obtained by a primary  $K_{2,2g}$ -contraction at  $C$ . Then  $G'$  is also essentially 3-edge connected by Fact 8. On the other hand, when (ii) holds in Lemma 9, there exists a subset  $Y_1 \subset Y$  with  $Y_1 \neq \emptyset$  and  $Y_1 \neq Y$  such that the graph  $G'$  obtained by a secondary  $K_{2,2g}$ -contraction at  $C$  with respect to  $Y_1$  is also essentially 3-edge connected. Note that in either case,  $|V(G')| < |V(G)|$ , so by the induction hypothesis,  $G'$  has a  $D$ -system  $\mathcal{D}'$  such that  $|\mathcal{D}'| \leq \max\{\frac{2}{5}(\alpha'(G') + 1), 1\}$ . By Lemmas 11 (i)–(iii),  $G$  also has a  $D$ -system  $\mathcal{D}$  such that  $|\mathcal{D}| \leq |\mathcal{D}'| \leq \max\{\frac{2}{5}(\alpha'(G') + 1), 1\} \leq \max\{\frac{2}{5}(\alpha'(G) + 1), 1\}$ . Thus, we may assume that  $G$  has no bad  $K_{2,2g}$  for any  $g \geq 2$ .  $\square$

## 6.2 $C_5$ -Contractions and a strong $D$ -system

In this subsection, we contract subgraphs isomorphic to  $C_5$  which are bad in the following sense. For a subgraph  $C$  of  $G$  with  $C \simeq C_5$ ,  $C$  is called *normal* if  $C$  has a neighbor outside of  $C$  that has degree one or two in  $G$ ; otherwise  $C$  is *abnormal*. Now we consider the following contractions.

Let  $\mathcal{C}$  be a set of pairwise vertex-disjoint cycles  $C$  of  $G$  such that  $C$  is an abnormal  $C_5$ . Take such a set  $\mathcal{C}$  so that  $|\mathcal{C}|$  is as large as possible. Now we perform  $C_5$ -contractions of each  $C \in \mathcal{C}$  and let  $G_1$  be the resulting graph. By Fact 8,  $G_1$  is also essentially 3-edge connected (but  $G_1$  might have multiple edges). In addition, we repeat  $C_2$ - or  $C_3$ -contractions to  $G_1$  until there does not exist a subgraph isomorphic to  $C_2$  or  $C_3$ . Let  $G'_1$  be the graph obtained by these operations. Again by Fact 8,  $G'_1$  is also essentially 3-edge connected.

Let  $G''_1$  be the graph obtained from  $G'_1$  by removing all pendant edges, and suppressing all vertices of degree two in  $G'_1$ . Since  $G'_1$  is essentially 3-edge connected,  $G''_1$  is 3-edge connected. Thus, by Theorem 12,  $G''_1$  has a spanning even subgraph  $H''_1$  in which each component has at least  $\min\{5, |G''_1|\}$  vertices.

Let  $\mathcal{D}'_1$  be the set of circuits of  $G'_1$  corresponding to components of  $H''_1$ . In other words, for each  $D' \in \mathcal{D}'_1$ , there exists a component  $D''$  of  $H''_1$  such that  $D''$  is the

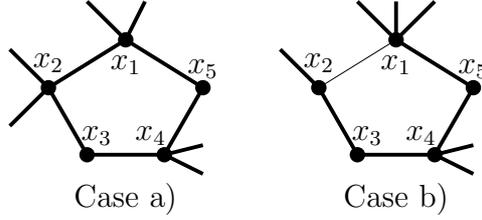


Figure 2: The circuit  $\tilde{D}$ .

circuit obtained from  $D'$  by suppressing all vertices of degree two in  $G'_1$ . Since  $G'_1$  is essentially 3-edge connected,  $\mathcal{D}'_1$  is a strong  $D$ -system in  $G'_1$ .

Next we consider reconstructions of  $C_2$ 's and  $C_3$ 's. By recursively applying Lemma 10 to  $\mathcal{D}'_1$  and  $G'_1$ , we obtain a strong  $D$ -system  $\mathcal{D}_1$  of  $G_1$ .

### 6.3 Reconstruction of good $C_5$ s and classification of bad $C_5$ s

Now we consider reconstructions of  $C_5$ s. Some vertices obtained by a contraction of a  $C_5$  could be reconstructed without increasing the number of circuits in  $\mathcal{D}_1$ . We call such a  $C_5$  *good*; otherwise it is a *bad*  $C_5$ . More precisely, we define a good  $C_5$  and a bad  $C_5$ , respectively, as follows.

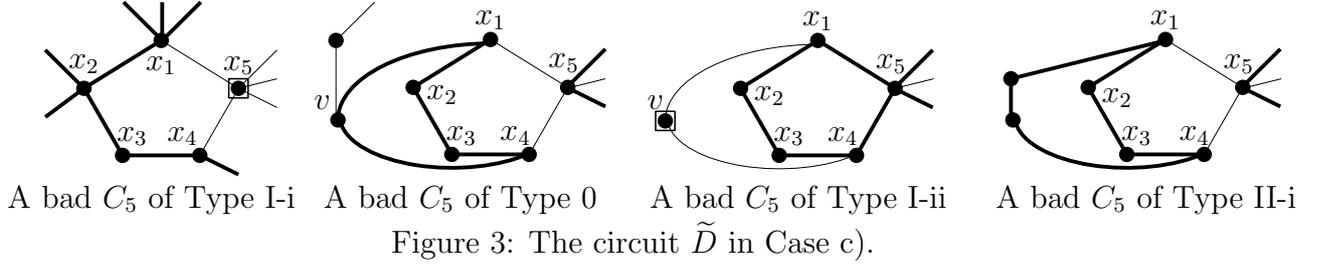
Let  $C = x_1x_2 \dots x_5 \in \mathcal{C}$  and let  $D$  be a circuit in  $\mathcal{D}_1$  which contains the vertex obtained by contraction of  $C$ . Now we regard  $D$  as the subgraph in  $G$  induced by all edges in  $D$ . Although  $x_i$ 's might have odd degree in  $D$ , all other vertices of  $D$  have even degree in  $D$ . Depending on the parities of degrees of  $x_i$ 's in  $D$ , we consider the following four cases:

- a) All  $x_i$ 's have even degrees in  $D$ .
- b) Two consecutive  $x_i$ 's have odd degrees and others have even.
- c) Exactly two  $x_i$ 's but not consecutive have odd degrees.
- d) Four  $x_i$ 's have odd degrees and the fifth one has even degree.

Note that in Cases a) and b), the following  $\tilde{D}$  is also a circuit in the graph obtained from  $G_1$  by reconstruction of  $C$ :

$$\tilde{D} := \begin{cases} D + E(C) & \text{if Case a) occurs,} \\ D + E(C) - \{x_1x_2\} & \text{if Case b) occurs and } x_1 \text{ and } x_2 \text{ have odd degrees.} \end{cases}$$

See Figure 2. Thus, in Case a) and b) we can reconstruct an abnormal  $C \simeq C_5$  without changing the number of circuits in  $\mathcal{D}_1$ . Note that all edges in  $G$  are dominated by  $(\mathcal{D}_1 - \{D\}) \cup \{\tilde{D}\}$ . Therefore, such a  $C_5$  is *good*.



Now we consider the remaining two cases. Let  $C = x_1 \dots x_5$  be an abnormal  $C \simeq C_5$  with Case c), and assume that  $x_1$  and  $x_4$  have odd degrees in  $D$  and others have even. In this case, we first consider the even subgraph

$$D^* := D + \{x_1x_2, x_2x_3, x_3x_4\}$$

of the graph obtained from  $G_1$  by reconstruction of  $C$ . If the degree of  $x_5$  in  $G$  is two, then letting  $\tilde{D} = D^*$ , we can reconstruct  $C$  without changing the number of circuits in  $\mathcal{D}_1$ . Therefore, such a  $C_5$  is *good*.

Now we assume that the degree of  $x_5$  in  $G$  is at least three. We also consider two cases depending on the degree of  $x_5$  in  $D^*$ . If  $x_5$  has degree zero in  $D$ , then we let such  $C$  be a *bad  $C_5$  of Type I-i*. See the left side of Figure 3. In this case, let  $\tilde{D} = D^*$ . Note that  $\tilde{D}$  does not pass through the vertex  $x_5$ . We call the vertex  $x_5$  *uncovered* and the two edges  $x_1x_5$  and  $x_4x_5$  *D-dominated edges by  $x_5$* . Next, we suppose that  $x_5$  has a degree at least two in  $D^*$ . If  $D^*$  has only one component, then we let  $\tilde{D} = D^*$  and we can use  $\tilde{D}$  as a circuit of  $G$ , so  $C$  is *good*; otherwise,  $C$  is *bad*.

Suppose  $C$  is bad in this sense. Then  $D^*$  has exactly two components such that one of them contains all vertices in  $V(C) \setminus \{x_5\}$  and the other contains  $x_5$ . Suppose further that the first one consists of only five vertices, say  $v, x_1, x_2, x_3$  and  $x_4$ , and  $v$  is not a contracted vertex from a bad  $C_5$ . If  $v$  is incident with a vertex outside of  $C$  and of degree one or two in  $G$ , then we call such  $C \simeq C_5$  a *bad  $C_5$  of Type 0*, say that the circuit  $vx_1x_2x_3x_4$  is *generated from  $C$* , and let  $\tilde{D} := D^*$ . See the left middle of Figure 3. Otherwise, that is, if  $v$  is not incident with a vertex outside of  $C$  and of degree one or two in  $G$ , then we call such  $C \simeq C_5$  a *bad  $C_5$  of Type I-ii*, and we let

$$\tilde{D} := D + E(C).$$

See the right middle of Figure 3. Moreover, we call the vertex  $v$  *uncovered*, and the two edges  $vx_1$  and  $vx_4$  are *D-dominated by  $v$* . For other case, that is, if the circuit containing  $V(C) - \{x_5\}$  has at least six vertices, or if  $v$  is a vertex contracted from a bad  $C_5$ , then we say that  $C$  is a *bad  $C_5$  of Type II-i*, and we let  $\tilde{D} = D^*$ . See the right side of Figure 3.

Finally, let  $C$  be an abnormal  $C_5$  with Case d), and assume that  $x_i$  has odd degree in  $D$  for all  $1 \leq i \leq 4$ . Then we consider the subgraph

$$D^* := D + \{x_2x_3, x_1x_5, x_4x_5\}$$

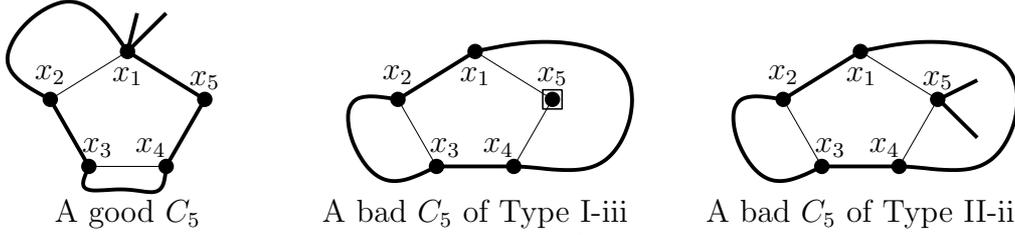


Figure 4: The circuit  $\tilde{D}$  in Case d).

of the graph obtained from  $G_1$  by reconstruction of  $C$ . If  $D^*$  has only one component, then we let  $\tilde{D} = D^*$  and we can use  $\tilde{D}$  as a circuit, so  $C$  is *good*. See the left side of Figure 4. Suppose now that  $D^*$  has two components such that one of them contains  $x_2$  and  $x_3$  and the other contains  $x_1, x_4$  and  $x_5$ . Then we consider the following subgraph

$$\tilde{D} := D + \{x_1x_2, x_3x_4\}$$

of the graph obtained from  $G_1$  by reconstruction of  $C$ . Since  $D^*$  has two components, there exist a path connecting  $x_2$  and  $x_3$  and a path connecting  $x_1$  and  $x_4$  in  $D$ . Therefore,  $\tilde{D}$  has at most two components. If the degree of  $x_5$  in  $G$  is two, then we can also use  $\tilde{D}$  as a circuit of  $G$ , so  $C$  is *good*. Suppose that the degree of  $x_5$  in  $G$  is at least three. Similarly to Case c), we consider two cases depending on the degree of  $x_5$  in  $\tilde{D}$ . When  $x_5$  has degree zero in  $\tilde{D}$ , then we call such  $C$  a *bad  $C_5$  of Type I-iii*. Also  $x_5$  is *uncovered* and the edges  $x_1x_5$  and  $x_4x_5$  are *D-dominated by  $x_5$* . See the middle of Figure 4. If  $x_5$  has degree at least two in  $\tilde{D}$  and  $\tilde{D}$  consists of two circuits, then such a  $C_5$  is said to be a *bad  $C_5$  of Type II-ii*; otherwise  $C$  is *good*. Note that when  $C$  is a bad  $C_5$  of Type II-ii,  $\tilde{D}$  has exactly two components such that one of them contains  $V(C) \setminus \{x_5\}$  and the other contains  $x_5$ . Notice also that the first one has at least six vertices. See the right side of Figure 4.

For an abnormal  $C \simeq C_5$ , we say that  $C$  is a *bad  $C_5$  of Type I* if  $C$  is of Type I-i or I-ii or I-iii, and  $C$  is a *bad  $C_5$  of Type II* if  $C$  is of Type II-i or II-ii.

In addition, for an abnormal  $C \simeq C_5$ , the operation to get  $\tilde{D}$  from  $D \in \mathcal{D}_1$  which contains the vertex contracted from  $C$  is also called *reconstruction (of  $C$ )*. Note that after reconstruction of all bad  $C_5$ , we obtain a set of circuits of  $G$  that dominates all edges in  $G$  except for those connecting two uncovered vertices. By the definition, we can reconstruct all good  $C_5$ s without increasing the number of circuits in  $\mathcal{D}_1$ .

Let  $G_2$  and  $\mathcal{D}_2$  be the graph and the  $D$ -system of  $G_2$  obtained from  $G_1$  and  $\mathcal{D}_1$  by reconstructing all good  $C_5$ s and all bad  $C_5$ s of Type 0. We call a circuit in  $\mathcal{D}_2$  that is not generated from a bad  $C_5$  of Type 0 *original*. Note that the set of original circuits in  $\mathcal{D}_2$  has a one-to-one correspondence to  $\mathcal{D}_1$ , also to  $\mathcal{D}'_1$ , since any generated circuit from a bad  $C_5$  of Type 0 corresponds to a subcircuit of a circuit in  $\mathcal{D}_1$  of length two, so it disappears after  $C_2$ -contraction. Notice also that  $\mathcal{D}_2$  is a strong  $D$ -system of  $G_2$ .

It is easy to show the following claim.

**Claim 3** *If  $\mathcal{D}_2$  has at least two original circuits, then each  $D \in \mathcal{D}_2$  with  $D \not\cong C_5$  has at least five vertices even after any sequence of  $C_2$ - and  $C_3$ -contractions and suppressing.*

**Proof.** Suppose that some circuit  $D \in \mathcal{D}_2$  with  $D \not\cong C_5$  has at most four vertices after a sequence of  $C_2$ - and  $C_3$ -contractions and suppressing. Since  $D \not\cong C_5$ ,  $D$  is not generated from a bad  $C_5$  of Type 0, and hence there exists a circuit  $D'$  in  $\mathcal{D}'_1$  that corresponds to  $D$ . Since  $D$  has at most four vertices after a sequence of  $C_2$ - and  $C_3$ -contractions and suppressing,  $D'$  also has at most four vertices. Since  $D'$  corresponds to some component of  $H''_1$ ,  $D'$  has at least  $\min\{5, |G''_1|\}$  vertices. This implies that  $|\mathcal{D}_1| = |\mathcal{D}'_1| = \omega(H''_1) = 1$ , where  $\omega(H''_1)$  is the number of components of  $H''_1$ . Hence  $\mathcal{D}_1$  has only one circuit, which implies that  $\mathcal{D}_2$  has exactly one original circuit.  $\square$

## 6.4 Reconstruction of bad $C_5$ s of Type II and construction of a matching

Let  $G_3$  be the graph obtained from  $G_2$  by reconstructing all bad  $C_5$ s of Type II. Recursively applying the reconstructions in Section 6.3, we get a strong  $D$ -system  $\mathcal{D}_3$  of  $G_3$ . However,  $|\mathcal{D}_3|$  might be larger than  $|\mathcal{D}_2|$ . In this subsection, we will show the existence of a matching in  $G_3$  having many edges comparing with the number of circuits in  $\mathcal{D}_3$ . (Actually, we will show Claim 4.)

For a circuit  $F \in \mathcal{D}_3$  and a matching  $M$  in  $G_3$ , we call  $F$  *special for  $M$*  if  $F$  contains a contracted vertex  $u$  from a bad  $C_5$  of Type I and (i) no edge incident with  $u$  is contained in  $M$ , or (ii)  $F \simeq K_{2,2g}$  for some  $g \geq 2$ ,  $u$  has degree  $2g$  in  $F$ , and  $|E(F) \cap M| = 1$ . For a matching  $M$  in  $G_3$ , we define the function  $f_M$  from  $\mathcal{D}_3$  to  $\{\frac{3}{2}, 2, \frac{5}{2}\}$  as follows; for every circuit  $F$  in  $\mathcal{D}_3$ ,

$$f_M(F) = \begin{cases} \frac{3}{2} & \text{if } F \text{ is special for } M, \\ 2 & \text{if } F \text{ contains a contracted vertex from a bad } C_5 \text{ of Type I} \\ & \text{and } F \text{ is not special for } M, \\ \frac{5}{2} & \text{otherwise.} \end{cases}$$

**Claim 4** *There exists a matching  $M_3$  in  $G_3$  such that*

$$\sum_{F \in \mathcal{D}_3} f_{M_3}(F) \leq |\mathcal{D}_3| + 1.$$

**Proof.** Let

$$\mathcal{D}_2^1 := \{D \in \mathcal{D}_2 : D \text{ contains no contracted vertex from a bad } C_5 \text{ of Type II}\},$$

and  $\mathcal{D}_2^2 := \mathcal{D}_2 - \mathcal{D}_2^1$ .

We define the mapping  $f$  from  $\mathcal{D}_3$  to  $\{2, \frac{5}{2}\}$  as follows; for every circuit  $F$  in  $\mathcal{D}_3$ ,

$$f(F) = \begin{cases} 2 & \text{if } F \text{ contains a contracted vertex from a bad } C_5 \text{ of Type I,} \\ \frac{5}{2} & \text{otherwise.} \end{cases}$$

Note that  $f_M(F) \leq f(F)$  for each matching  $M$  of  $G_3$  and each circuit  $F \in \mathcal{D}_3$ .

Notice that every circuit  $D$  in  $\mathcal{D}_2^1$  is also a circuit in  $\mathcal{D}_3$ . For all  $D \in \mathcal{D}_2^1$ , let  $G_D$  be the subgraph of  $G_3$  induced by  $V(D) \cup \{v \in N_{G_3}(D) : d_{G_3}(v) = d_G(v) = 1 \text{ or } 2\}$ . We will show the following subclaim.

**Subclaim 1** *For every  $D \in \mathcal{D}_2^1$ , there is a matching  $M_D$  in  $G_D$  with at least  $f(D) - 1$  edges such that  $\bigcup_{D \in \mathcal{D}_2^1} M_D$  is also a matching in  $G_3$ , and*

$$\sum_{D \in \mathcal{D}_2^1} |M_D| \geq \begin{cases} \sum_{D \in \mathcal{D}_2^1} f(D) - 1 & \text{if } \mathcal{D}_2 \text{ has only one original circuit and} \\ & \text{that circuit lies in } \mathcal{D}_2^1, \\ \sum_{D \in \mathcal{D}_2^1} f(D) & \text{otherwise.} \end{cases}$$

On the other hand, let  $D \in \mathcal{D}_2^2$ . Note that  $D$  is divided into more than one circuit through reconstructions of bad  $C_5$ s of Type II. Let  $\mathcal{F}_D$  be the set of circuits  $F$  in  $\mathcal{D}_3$  such that  $E(F) \cap E(D) \neq \emptyset$ . Note that  $\mathcal{D}_3 = \mathcal{D}_2^1 \cup \bigcup_{D \in \mathcal{D}_2^2} \mathcal{F}_D$ . We will also show the following.

**Subclaim 2** *For every  $D \in \mathcal{D}_2^2$ , there is a matching  $M_D$  in  $G_3[\bigcup_{F \in \mathcal{F}_D} V(F)]$  such that*

$$|M_D| \geq \begin{cases} \sum_{F \in \mathcal{F}_D} f_{M_D}(F) - 1 & \text{if } D \text{ is the only original circuit in } \mathcal{D}_2, \\ \sum_{F \in \mathcal{F}_D} f_{M_D}(F) & \text{otherwise.} \end{cases}$$

Suppose that both Subclaims 1 and 2 hold. Then  $M_3 := \bigcup_{D \in \mathcal{D}_2} M_D$  is a matching in  $G_3$ . Moreover, since the first case in the inequality in Subclaim 1 and the first case in the inequality in Subclaim 2 do not occur at the same time, we have

$$\begin{aligned} |M_3| &\geq \sum_{D \in \mathcal{D}_2^1} |M_D| + \sum_{D \in \mathcal{D}_2^2} |M_D| \\ &\geq \sum_{D \in \mathcal{D}_2^1} f(D) + \sum_{D \in \mathcal{D}_2^2} \sum_{F \in \mathcal{F}_D} f_{M_D}(F) - 1 \\ &\geq \sum_{F \in \mathcal{D}_3} f_{M_3}(F) - 1, \end{aligned}$$

which completes the proof of Claim 4. Therefore, it suffices to prove Subclaims 1 and 2.

**Proof of Subclaim 1.** Recall that for  $D \in \mathcal{D}_2^1$ ,  $D$  is also a circuit in  $\mathcal{D}_3$  since  $D$  has no contracted vertex from a bad  $C_5$  of Type II. Recall also that  $G_D$  is the subgraph of  $G_3$  induced by  $V(D) \cup \{v \in N_{G_3}(D) : d_{G_3}(v) = d_G(v) = 1 \text{ or } 2\}$ .

We will first show that for all  $D \in \mathcal{D}_2^1$  with  $D \not\cong C_5$ , there is a matching  $M_D$  in  $G_D$  with  $|M_D| \geq f(D) - 1$  if  $D$  is the only original circuit in  $\mathcal{D}_2$ , and  $|M_D| \geq f(D)$  otherwise. Let  $D \in \mathcal{D}_2^1$  with  $D \not\cong C_5$ .

Suppose first that  $D$  is the only original circuit in  $\mathcal{D}_2$ . If  $D$  contains a contracted vertex from a bad  $C_5$  of Type I, then  $D \subset G_D$  contains a matching  $M_D$  with  $|M_D| = 1 = f(D) - 1$ . On the other hand, if  $D$  contains no contracted vertex from a bad  $C_5$  of Type I, then  $D$  is also a circuit in  $G$ , and hence  $D$  is not a flower by Claim 1. Then we can find a matching  $M_D$  in  $D$  with  $|M_D| = 2 > f(D) - 1$  by Lemma 14 (i). So we may assume that  $\mathcal{D}_2$  has at least two original circuits.

By Claim 3,  $D$  has at least five vertices and we can find a matching with two edges in  $D$  by Lemma 14 (ii). Hence if  $D$  has a contracted vertex from a bad  $C_5$  of Type I, then we can find a matching  $M_D$  with at least  $f(D)$  edges in  $D$ , and we are done. So we may assume that  $D$  has no contracted vertex from a bad  $C_5$  of Type I, and hence  $D$  is also a circuit in  $G$ . Hence  $D$  contains no cycle of length two or three by Claim 1. By Lemma 14 (iii), if  $D \not\cong K_{2,2g}$  with  $g \geq 2$ , then we can find a matching with three edges, and we are done. (Recall that  $D \not\cong C_5$ .) So we may also assume that  $D \simeq K_{2,2g}$  with  $g \geq 2$ . By Claim 2,  $D$  is good. Thus,  $D$  has only at most four vertices of degree at least three in  $G$ . This implies that after suppressing all vertices of degree two,  $D$  has only at most four vertices, but this contradicts Claim 3. Thus, for all  $D \in \mathcal{D}_2^1$  with  $D \not\cong C_5$ , there is a matching  $M_D$  in  $G_D$  with  $|M_D| \geq f(D) - 1$  if  $D$  is the only original circuit in  $\mathcal{D}_2$ , and  $|M_D| \geq f(D)$  otherwise.

We next consider all  $D \in \mathcal{D}_2^1$  that are isomorphic to  $C_5$ . Note that  $D$  has no contracted vertex from a bad  $C_5$  of Type II. If  $D$  is generated from a bad  $C_5$  of Type 0, then by the definition,  $D$  has a neighbor of degree one or two in  $G$ . Otherwise  $D$  is vertex-disjoint from any  $C \in \mathcal{C}$ . Therefore, if  $D$  is abnormal, this contradicts the maximality of  $|\mathcal{C}|$ . Thus,  $D$  is normal. In either cases,  $D$  has a neighbor of degree one or two in  $G$ .

If  $D$  has a neighbor of degree one, then by Fact 13, there exist three edges forming a matching in  $G_D - \bigcup_{D' \in \mathcal{D}_2^1 \setminus \{D\}} V(G_{D'})$  and we are done. Therefore, it suffices to consider only the set, say  $\mathcal{C}_2$ , of circuits  $D$  in  $\mathcal{D}_2^1$  such that  $D \simeq C_5$ ,  $D$  has no vertex contracted from a bad  $C_5$  and  $D$  is adjacent with a vertex of degree two.

Let  $R$  be the bipartite graph such that one vertex set of the bipartition of  $R$  is  $\mathcal{C}_2$ , the other one is the set of vertices of degree two in  $G_3$ , and  $D \in \mathcal{C}_2$  is adjacent with  $v$  in  $R$  if and only if  $v$  is adjacent with a vertex of  $D$  in  $G_3$ . By the definition, each  $D \in \mathcal{C}_2$  has a degree at least one in  $R$ . Let  $R'$  be a component of  $R$  containing at least one vertex in  $\mathcal{C}_2$ . If  $R'$  has only one vertex in  $\mathcal{C}_2$ , say  $D \in \mathcal{C}_2$ , then  $D \cup \{\varphi(D)\}$  has a matching in  $G_3$  with three edges, where  $\varphi(D)$  is a vertex of degree two in  $G_3$ .

which is a neighbor of  $D$  in  $R$ . (Note that the matching is also in  $G_D$ .) So we may assume that  $|\mathcal{C}_2 \cap V(R')| \geq 2$ , and let  $\vec{T}$  be a rooted spanning tree of  $R'$  with root  $D^*$  for some  $D^* \in \mathcal{C}_2$ . Since each  $D \in \mathcal{C}_2$  has a vertex incident with a vertex of degree two in  $G_3$ , each  $D \in \mathcal{C}_2 \cap V(R')$  has a parent  $\varphi(D)$  in  $\vec{T}$ , except for  $D = D^*$ . Let  $\varphi(D^*) = \emptyset$ . By Fact 13,  $D \cup \{\varphi(D)\}$  has a matching  $M_D$  in  $G_3$  with three edges for each  $D \in \mathcal{C}_2 \cap V(R')$  with  $D \neq D^*$ , and with two edges for  $D = D^*$ . Then

$$\begin{aligned}
\sum_{D \in \mathcal{C}_2 \cap V(R')} |M_D| &\geq 3(|\mathcal{C}_2 \cap V(R')| - 1) + 2 \\
&= 3|\mathcal{C}_2 \cap V(R')| - 1 \\
&= \frac{5}{2}|\mathcal{C}_2 \cap V(R')| + \frac{1}{2}|\mathcal{C}_2 \cap V(R')| - 1 \\
&\geq \sum_{D \in \mathcal{C}_2 \cap V(R')} f(D).
\end{aligned}$$

Considering all components of  $R$ , this completes the proof of Subclaim 1.  $\square$

**Proof of Subclaim 2.** Let  $D \in \mathcal{D}_2^2$ . Recall that  $\mathcal{F}_D$  is the set of circuits  $F$  in  $\mathcal{D}_3$  such that  $E(F) \cap E(D) \neq \emptyset$ . For a circuit  $F \in \mathcal{F}_D$ , let  $D_F$  be the subcircuit of  $D$  such that  $E(D_F) = E(F) \cap E(G_2)$ .

By the definition, each contracted vertex from a bad  $C_5$  of Type II is a cut vertex of  $D$  (otherwise we can reconstruct such a vertex without increasing the number of circuits, so it is good). Therefore  $D$  has a tree-like structure. More precisely, let  $T$  be the graph such that the vertex set of  $T$  is  $\mathcal{F}_D$  and two vertices  $F$  and  $F'$  are joined by an edge in  $T$  if and only if  $D_F$  and  $D_{F'}$  share a contracted vertex from a bad  $C_5$  of Type II. Note that  $T$  is a tree.

Let  $F \in \mathcal{F}_D$  be a leaf of  $T$ . Note that  $D_F$  has exactly one contracted vertex from a bad  $C_5$  of Type II, say  $u$ . Let  $C = x_1x_2 \dots x_5$  be the bad  $C_5$  in  $G_3$  corresponding to  $u$ . Suppose that  $D_F$  has only two vertices, and let  $v$  be the (only) vertex in  $V(D_F) \setminus \{u\}$ . If  $|F| = 2$ , then  $v$  is a contracted vertex from a bad  $C_5$  of Type I since  $G$  is simple. Otherwise, that is, if  $|F| > 2$ , then  $C$  is a bad  $C_5$  of Type II-i by the definition, and we may assume that  $F$  consists of five vertices  $x_1, x_2, x_3, x_4$  and  $v$ . Then, by the definition of Type II-i,  $v$  has to be a contracted vertex from a bad  $C_5$  of Type I. This implies that if  $|D_F| = 2$ , then  $F$  contains a contracted vertex from a bad  $C_5$  of Type I.

Let  $L$  be the set of leaves of  $T$ , and let  $L' \subset L$  be the set of circuits  $F \in \mathcal{F}_D$  such that  $D_F$  contains at least three vertices. By the above fact, each component in  $L \setminus L'$  has a contracted vertex from a bad  $C_5$  of Type I, and hence at least  $|L \setminus L'|$  circuits in  $\mathcal{F}_D$  contain a contracted vertex from a bad  $C_5$  of Type I. Thus, for every

matching  $M$  in  $G_3$ ,

$$\begin{aligned}
\sum_{F \in \mathcal{F}_D} f_M(F) &\leq \sum_{F \in \mathcal{F}_D} f(F) \\
&\leq \frac{5}{2}(|T| - |L \setminus L'|) + 2|L - L'| \\
&= \frac{5}{2}|T| - \frac{1}{2}|L| + \frac{1}{2}|L'|. \tag{1}
\end{aligned}$$

Let  $T' = T - L'$  and let  $I$  be a maximum independent set of  $T'$ . Note that  $|I| \geq \frac{1}{2}|T'|$ , since  $T'$  is bipartite. Taking one edge from  $D_F$  for each  $F \in I$ , we can find a matching  $M'$  in  $D$  of order at least  $|I|$ . Moreover, for each  $F \in L'$ ,  $D_F$  has an edge which is not incident with the vertex  $u$ , where  $u$  is the unique contracted vertex in  $D_F$  from a bad  $C_5$  of Type II. Therefore, in  $D$ , we can find a matching  $\widetilde{M}$  with at least  $|I| + |L'|$  edges.

Let  $u$  be a vertex in  $D$  that contracted from a bad  $C_5$  of Type II, and let  $C$  be the bad  $C_5$  in  $G_3$  corresponding to  $u$ . By Fact 13, for each edge  $e$  in  $D$  incident with  $u$ , we can find two edges in  $C$  which together with  $e$  form a matching in  $G_3$ . This implies that for each contracted vertex in  $D$  from  $C$  that is a bad  $C_5$  of Type II, we can add two edges into the matching  $\widetilde{M}$  through the reconstruction of  $C$ . Since  $D$  has  $|T| - 1$  contracted vertices from bad  $C_5$ s of Type II, there exists a matching  $M_D$  in  $G_3[\bigcup_{F \in \mathcal{F}_D} V(F)]$  such that

$$\begin{aligned}
|M_D| &\geq |I| + |L'| + 2(|T| - 1) \\
&\geq \frac{1}{2}|T'| + |L'| + 2|T| - 2 \\
&= \frac{5}{2}|T| + \frac{1}{2}|L'| - 2.
\end{aligned}$$

If  $D$  is the only original circuit in  $\mathcal{D}_2$ , then by the inequality (1),

$$\begin{aligned}
|M_D| &\geq \frac{5}{2}|T| + \frac{1}{2}|L'| - 2 \\
&\geq \frac{5}{2}|T| + \frac{1}{2}|L'| - \frac{1}{2}|L| - 1 \\
&\geq \sum_{F \in \mathcal{F}_D} f_{M_D}(F) - 1,
\end{aligned}$$

and hence  $M_D$  is a desired matching. On the other hand, if  $|L| \geq 4$ , then

$$\begin{aligned}
|M_D| &\geq \frac{5}{2}|T| + \frac{1}{2}|L'| - 2 \\
&\geq \frac{5}{2}|T| + \frac{1}{2}|L'| - \frac{1}{2}|L| \\
&\geq \sum_{F \in \mathcal{F}_D} f_{M_D}(F),
\end{aligned}$$

and we are also done. Thus, we may assume that  $\mathcal{D}_2$  has at least two original circuits and  $|L| \leq 3$ .

Moreover, we may also assume that

(T0)  $D$  has no matching with at least  $\frac{1}{2}|T'| + |L'| + 1$  edges, and if  $|L| = 3$ , then  $D$  has no matching with at least  $\frac{1}{2}|T'| + |L'| + \frac{1}{2}$  edges.

This also implies the following facts.

(T1)  $T'$  has no independent set of order at least  $\frac{1}{2}|T'| + 1$ .

(T2) If  $|L| = 3$ , then  $T'$  has no independent set of order at least  $\frac{1}{2}(|T'| + 1)$ , that is,  $T'$  is a balanced bipartite graph.

(T3) For each  $F \in L'$ ,  $D_F - \{u\}$  has no matching with at least two edges, where  $u$  is the unique contracted vertex from a bad  $C_5$  of Type II.

Suppose first that  $|L| = 3$ . Let  $F^* \in \mathcal{F}_D$  such that the degree of  $F^*$  in  $T$  is exactly 3. Note that  $D_{F^*}$  has at least three vertices. Let  $T^1, T^2$  and  $T^3$  be the three paths in  $T' - F^*$  (possibly  $T^i = \emptyset$  for some  $i$ , which could happen when  $F^*$  is adjacent with a member of  $L'$  in  $T$ ). By (T2),  $T'$  is a balanced bipartite graph, and hence at least one of the paths  $T^1, T^2, T^3$ , say,  $T^1$ , has odd number of vertices. Since  $T' - T^1$  is a path of odd order,  $T' - T^1$  has an independent set  $I_0$  with  $|I_0| \geq \frac{1}{2}|T' - T^1| + \frac{1}{2}$ . Similarly,  $T^1$  has an independent set  $I_1$  with  $|I_1| \geq \frac{1}{2}|T^1| + \frac{1}{2}$ . Since  $D_{F^*}$  has at least three vertices, even if  $F^* \in I_0$ , we can take an edge from  $D_F$  for each  $F \in I_0 \cup I_1$  so that such edges form a matching  $M'$  in  $D$ . Adding one edge from  $D_F$  for each  $F \in L'$ , we can obtain a matching in  $D$  with

$$|M'| + |L'| = |I_0| + |I_1| + |L'| \geq \frac{1}{2}|T'| + |L'| + 1$$

edges, contradicting (T0). Thus, we may assume that  $|L| = 2$ , that is,  $T$  is a path. Let  $T = F^1 F^2 \dots F^l$ .

A circuit is called *redundant* if it is reduced to one vertex by a sequence of  $C_2$ - or  $C_3$ -contractions. If for all circuits  $F \in \mathcal{F}_D$ ,  $D_F$  is redundant, then  $D$  is also redundant, contradicting Claim 3 and the fact that  $|\mathcal{D}_2| \geq 2$ . Hence there exists a circuit  $F^* \in \mathcal{F}_D$  such that  $D_{F^*}$  is not redundant.

Suppose also that there exists a circuit  $F^{**} \in \mathcal{F}_D$  such that  $F^{**} \neq F^*$  and  $D_{F^{**}}$  is not redundant. We may assume that  $F^* = F^i$  and  $F^{**} = F^j$  for some  $i < j$ . Let  $T^1 = F^1 \dots F^i$ ,  $T^2 = F^i \dots F^j$  and  $T^3 = F^j \dots F^l$ . If  $|T^1|$  is odd, then we can find a matching in  $\bigcup_{F \in V(T^1)} D_F$  with at least  $\frac{1}{2}(|T^1| + 1)$  edges. On the other hand, if  $|T^1|$  is even, then we can find a matching in  $\bigcup_{F \in V(T^1)} D_F$  with at least  $\frac{1}{2}|T^1|$  edges if  $F^1 \notin L'$ , and with at least  $\frac{1}{2}|T^1| + 1$  edges if  $F^1 \in L'$ . In either case,  $\bigcup_{F \in V(T^1)} D_F$  has a matching  $M^1$  with at least  $\frac{1}{2}(|T^1| + |L' \cap T^1|)$  edges. Similarly,  $\bigcup_{F \in V(T^2)} D_F$

and  $\bigcup_{F \in V(T^3)} D_F$  have matchings  $M^2$  and  $M^3$  with at least  $\frac{1}{2}|T^2|$  edges and with at least  $\frac{1}{2}(|T^3| + |L' \cap T^3|)$  edges, respectively. Since both  $F^i$  and  $F^j$  are not redundant, both have a cycle of length at least 4. Therefore, we can take such matchings  $M^1$ ,  $M^2$  and  $M^3$  so that  $M^1 \cup M^2 \cup M^3$  is also a matching with  $|M^1| + |M^2| + |M^3|$  edges. Therefore,  $G_3[\bigcup_{F \in \mathcal{F}_D} V(F)]$  has a matching with at least

$$\begin{aligned} & \frac{1}{2}(|T^1| + |L' \cap T^1|) + \frac{1}{2}|T^2| + \frac{1}{2}(|T^3| + |L' \cap T^3|) + 2(|T| - 1) \\ &= \frac{1}{2}(|T| + 2 + |L'|) + 2|T| - 2 \\ &= \frac{5}{2}|T| + \frac{1}{2}|L'| - 1 \\ &= \frac{5}{2}|T| + \frac{1}{2}|L'| - \frac{1}{2}|L| \end{aligned}$$

edges, and we are done by the inequality (1). Thus, we may also assume that  $F^*$  is the only circuit in  $\mathcal{F}_D$  such that  $D_{F^*}$  is not redundant. Note that after  $C_2$ - or  $C_3$ -contractions and suppressing all vertices of degree 2,  $D_{F^*}$  has at least five vertices by Claim 3.

Suppose that  $D_{F^*} \not\cong K_{2,2g}$  for any  $g \geq 2$ . Then by Lemma 14 (ii) and by (T3),  $D_{F^*} \not\subseteq L'$ . Recall that  $I$  is a maximum independent set of  $T'$ . Then  $|I| \geq \frac{1}{2}|T'|$ . We can take two edges from  $D_{F^*}$  when  $F^* \in I$  and one edge from  $D_{F^*}$  when  $F^* \notin I$ , such that they, together with an edge in  $D_F$  for each  $F \in I$ , form a matching. This implies that there exists a matching in  $D$  with at least  $\frac{1}{2}|T'| + |L'| + 1$  edges, contradicting (T0). Thus, we obtain that  $D_{F^*} \cong K_{2,2g}$  for some  $g \geq 2$ .

Let  $F, F'$  be circuits in  $\mathcal{F}_D$  such that  $D_F$  and  $D_{F'}$  share a contracted vertex, say  $u$ , from a bad  $C_5$  of Type II. By the definition of the reconstruction of a bad  $C_5$  of Type II, exactly one of  $D_F$  and  $D_{F'}$  does not change through the reconstruction of  $C_u$ , where  $C_u$  is the bad  $C_5$  of Type II corresponding to  $u$ . Since  $D$  has  $|T| - 1$  contracted vertices from bad  $C_5$ s of Type II, at least one circuit in  $\mathcal{F}_D$ , say  $F$ , is also a circuit in  $D$ . So,  $F = D_F$ . Since  $F^*$  is the unique circuit in  $\mathcal{F}_D$  such that  $D_{F^*}$  is not redundant, we have that  $F = D_F$  is redundant or  $F = F^* \cong K_{2,2g}$  for some  $g \geq 2$ .

If  $F = F^*$  and  $F$  contains no contracted vertex from a bad  $C_5$  of Type I, then  $F$  is also a circuit in  $G$ . Then by Claim 2,  $F$  is a good  $K_{2,2g}$ . However, after  $C_2$ - and  $C_3$ -contractions and suppressing all vertices of degree 2,  $D$  has only at most four vertices in  $G''_1$ , contradicting Claim 3 and the fact  $|\mathcal{D}| \geq 2$ . Thus, if  $F = F^*$ , then  $F$  contains a contracted vertex from a bad  $C_5$  of Type I. On the other hand, even when  $F$  is redundant,  $F$  contains a contracted vertex from a bad  $C_5$  of Type I, since  $G$  is simple and triangle-free by Claim 1. In either case,  $F$  contains a contracted vertex from a bad  $C_5$  of Type I, say  $u$ .

We will show that

$$\begin{aligned}
& \text{there exists a matching } \widetilde{M} \text{ in } D \text{ with at least } \frac{1}{2}|T'| + |L'| + \frac{1}{2} \text{ edges} \\
& \text{such that (i) } \widetilde{M} \text{ does not contain any edge incident with } u \text{ or} \tag{2} \\
& \text{(ii) } F = F^* \simeq K_{2,2g} \text{ and } u \text{ has degree } 2g \text{ in } F \text{ and } |E(\widetilde{M}) \cap E(F)| = 1.
\end{aligned}$$

This implies that after reconstructing all contracted vertices in  $D$  from bad  $C_5$ s of Type II, adding some edges into  $\widetilde{M}$ , we can find a matching  $M_D$  in  $G_3[\bigcup_{F' \in \mathcal{F}_D} V(F')]$  with at least

$$\begin{aligned}
\frac{1}{2}|T'| + |L'| + \frac{1}{2} + 2(|T| - 1) &= \frac{5}{2}|T| + \frac{1}{2}|L'| - \frac{3}{2} \\
&= \frac{5}{2}(|T| - |L \setminus L'|) + 2(|L \setminus L'| - 1) + \frac{3}{2}
\end{aligned}$$

edges. (Recall that  $|L| = 2$ .) By the choice (2) and the definition,  $F$  is special for  $M_D$ , and hence  $\sum_{F' \in \mathcal{F}_D} f_{M_D}(F') \leq \frac{5}{2}(|T| - |L \setminus L'|) + 2(|L \setminus L'| - 1) + \frac{3}{2}$ . This completes the proof of Subclaim 2.

In the rest of the proof of Subclaim 2, we will show (2). Recall that  $T = F^1 F^2 \dots F^l$ . We may assume that  $F = F^l$  if  $F \in L'$ . Let  $F^i = F^*$ . Let  $T^1 = F^1 \dots F^i$  if  $F^1 \notin L'$ ; otherwise let  $T^1 = F^2 \dots F^i$ . Similarly, let  $T^2 = F^i \dots F^l$  if  $F^l \notin L'$ ; otherwise  $T^2 = F^i \dots F^{l-1}$ . Note that if  $F^* = F^i = F^l$ , then  $T^2 = \emptyset$ .

Since both  $T^1$  and  $T^2$  is a path, for  $j = 1, 2$ , we can find a matching  $M^j$  in  $\bigcup_{F' \in V(T^j)} D_{F'}$  with at least  $\frac{1}{2}|T^j|$  edges. Since  $D_{F^i} = D_{F^*}$  is not redundant,  $D_{F^i}$  has a cycle of length at least 4, and hence we can choose  $M^1$  and  $M^2$  such that  $M^1 \cup M^2$  is also a matching with  $|M^1| + |M^2|$  edges. Since we can take an edge from each circuit  $F'$  in  $L'$  which is not incident with the contracted vertex from bad  $C_5$  from Type II, there exists a matching in  $D$  with

$$\begin{aligned}
|M^1| + |M^2| + |L'| &\geq \frac{1}{2}|T^1| + \frac{1}{2}|T^2| + |L'| \\
&= \frac{1}{2}|T'| + |L'| + \frac{1}{2}
\end{aligned}$$

edges. Moreover, if  $|M^1| \geq \frac{1}{2}|T^1| + \frac{1}{2}$ , then together with  $M^2$ , it forms a matching of  $D$  with at least  $\frac{1}{2}|T^1| + \frac{1}{2} + \frac{1}{2}|T^2| \geq \frac{1}{2}|T'| + 1$  edges, contradicting (T1). Thus, we have that  $|M^1| = \frac{1}{2}|T^1|$ , that is,  $T^1$  has even number of vertices. Similarly,  $T^2$  also has even number of vertices.

Therefore, if  $F \in \mathcal{F}_D \setminus L'$ , then we can choose  $M^1$  and  $M^2$  such that every edge in  $F$  is not used in  $M^1 \cup M^2$ . This together with appropriate edges in  $D_{F^1}$  (if  $F^1 \in L'$ ) and in  $D_{F^l}$  (if  $F^l \in L'$ ) forms a matching  $\widetilde{M}$  in  $D$ , which is a desired one in (2)-(i).

So we may assume that  $F \in L'$ . Suppose first that  $F = F^1$ . Note that  $F \neq F^*$  by the choice of  $F^1$ . Since  $|T^1|$  is even, we can choose a matching  $M^1$  such that any

edge in  $D_{F^2}$  is not used in  $M^1$ . Then there exists an edge  $e$  in  $D_{F^1}$  which is not incident with the vertex  $u$ , since  $D_{F^1}$  has at least three vertices. Therefore,  $M^1 \cup \{e\}$ , together with  $M^2$  (and an edge in a circuit in  $D_{F^l}$  if  $F^l \in L'$ ), forms a matching  $\widetilde{M}$  in  $D$ , which is a desired one for (2)-(i). When  $F = F^l$  but  $F \neq F^*$ , or when  $F = F^l = F^*$  and the degree of  $u$  in  $D_{F^*}$  is 2, then similarly we can find a desired matching  $\widetilde{M}$  in  $D$  for (2)-(i). So we may assume that  $F = F^l = F^*$  and the degree of  $u$  in  $D_{F^*}$  is  $2g$ . In this case,  $M^1 \cup M^2$  together with an appropriate edge in  $D_{F^1}$  (if  $F^1 \in L'$ ) and an edge incident with  $u$  in  $F^l$  forms a matching  $\widetilde{M}$  in  $D$ , which is a desired one for (2)-(ii). This completes the proofs of Subclaim 2 and Claim 4.  $\square$

## 6.5 Reconstruction of bad $C_5$ s of Type I

In this subsection, we reconstruct all bad  $C_5$ s of Type I in  $G_3$ . After reconstructing all such bad  $C_5$ s, we get the original graph  $G$ , and we also get a set of circuits of  $G$  from  $\mathcal{D}_3$ , say  $\widetilde{\mathcal{D}}$ . Note that  $\widetilde{\mathcal{D}}$  might not be a  $D$ -system of  $G$ , because some edges incident with uncovered vertices might be not dominated by any circuit in  $\widetilde{\mathcal{D}}$ . In order to dominate all such edges, we shall add some circuits and some stars with centers at uncovered vertices. In this process, the number of members in the  $D$ -system increases, but we will show that we do not need to add too many circuits and stars.

Let  $K$  be the subgraph of  $G$  induced by the set of uncovered vertices. Note that any edge of  $K$  is not dominated by any circuit in  $\widetilde{\mathcal{D}}$ . For an uncovered vertex  $v$  contained in a bad  $C_5$  of Type I, say  $C$ , we call  $v$  *special* if  $F$  is special for  $M_3$ , where  $F$  is the circuit in  $\mathcal{D}_3$  passing the vertex corresponding to  $C$ . An uncovered vertex  $v$  is *non-special* if  $v$  is not special for  $M_3$ .

Let  $C^1, C^2, \dots, C^l$  be vertex disjoint cycles in  $K$ . Taking as many such cycles as possible, we can assume that  $K'$  has no cycle, where  $K' = K - \bigcup_{i=1}^l V(C^i)$ . Let  $V_0^S$  and  $V_0^N$  be the set of special vertices and the set of non-special vertices in  $\bigcup_{i=1}^l V(C^i)$ , respectively. Since  $G$  is simple and triangle-free, for all  $1 \leq i \leq l$ ,  $C^i$  has at least four vertices, and hence

$$l \leq \frac{1}{4} \left| \bigcup_{i=1}^l V(C^i) \right| = \frac{1}{4} (|V_0^S| + |V_0^N|). \quad (3)$$

Taking a smaller partite set of each component of  $K'$ , we obtain an independent set  $I$  of  $K'$  which dominates all edges in  $K'$ . Thus, there exists a mapping  $\psi$  from  $E(K')$  to  $I$  such that for all  $e \in E(K')$ ,  $e$  is incident with  $\psi(e) \in I$ . Note that  $I$  does not contain an isolated vertex in  $K'$ , and hence  $|\psi^{-1}(v)| \geq 1$  for each  $v \in I$ .

Let  $v \in I$ . Since  $v$  is uncovered with respect to some  $F \in \mathcal{D}_3$ , there exist two  $D$ -dominated edges by  $v$ . Let  $S_v$  be the star which is formed by a center  $v$  together with the edges in  $\psi^{-1}(v)$  and two  $D$ -dominated edges by  $v$ . In particular,  $S_v$  is a

star with at least three edges for all  $v \in I$ , and  $E(K') \subset \bigcup_{v \in I} E(S_v)$ . So,  $\mathcal{D}$  is a  $D$ -system of  $G$ , where

$$\mathcal{D} := \tilde{\mathcal{D}} \cup \{C^1, \dots, C^l\} \cup \{S_v : v \in I\}.$$

Let  $\mathcal{S}_1$  be the set of stars  $S'_v$  in  $\{S_v : v \in I\}$  such that  $S'_v$  contains an edge both of whose end vertices are special. Let  $\mathcal{S}_2 := \{S_v : v \in I\} - \mathcal{S}_1$ . For  $i = 1, 2$ , let  $V_i^S$  and  $V_i^N$  be the set of special vertices and the set of non-special vertices in  $\mathcal{S}_i$ , respectively. Since for all  $v \in I$ ,  $S_v$  contains at least two vertices in  $V(K')$ , we have that

$$|\mathcal{S}_1| \leq \frac{1}{2}(|V_1^S| + |V_1^N|) \quad \text{and} \quad |\mathcal{S}_2| \leq \frac{1}{2}(|V_2^S| + |V_2^N|). \quad (4)$$

On the other hand, since each star in  $\mathcal{S}_2$  has to contain a non-special vertex, we obtain

$$|\mathcal{S}_2| \leq |V_2^N|. \quad (5)$$

Let  $\mathcal{D}_3^S \subset \mathcal{D}_3$  be the set of special circuits for  $M_3$ , and let  $\mathcal{D}_3^N \subset \mathcal{D}_3$  be the set of circuits  $F$  in  $\mathcal{D}_3$  such that  $F$  contains a contracted vertex from a bad  $C_5$  of Type I and  $F$  is not special for  $M_3$ . Since every circuit in  $\mathcal{D}_3^S$  (and in  $\mathcal{D}_3^N$ ) corresponds to at least one special uncovered vertex for  $M_3$ , (one non-special uncovered vertex, respectively,) we obtain that

$$|\mathcal{D}_3^S| \leq |V_0^S| + |V_1^S| + |V_2^S|, \quad (6)$$

$$\text{and} \quad |\mathcal{D}_3^N| \leq |V_0^N| + |V_1^N| + |V_2^N|. \quad (7)$$

On the other hand, when we reconstruct each bad  $C_5$ , by Fact 13, we can find two edges which can be added into the matching  $M_3$  of  $G_3$ .

Moreover, let  $S_v \in \mathcal{S}_1$  and let  $vv'$  be an edge of  $S_v$  both of whose end vertices are special. Let  $C_v = x_1x_2 \dots x_5$  and  $C_{v'} = x'_1x'_2 \dots x'_5$  be the bad  $C_5$ s corresponding to  $v$  and  $v'$ , respectively. Let  $F_v$  be the circuit in  $\mathcal{D}_3$  containing  $v$ .

Suppose first that  $C_v$  is a bad  $C_5$ s of Type I-i or I-iii. In this case, by symmetry, we may assume that  $v = x_5$ . If all edges of  $F_v$  incident with  $x_i$  for  $1 \leq i \leq 5$  are not used in  $M_3$ , then let  $e_1 = x_1x_2$  and  $e_2 = x_3x_4$ . Otherwise,  $F_v \simeq K_{2,2g}$  for some  $g \geq 2$  since  $v$  is special. In this case, we may also assume that  $x_1$  is incident with an edge in  $M_3$ . Since  $C_v$  is a bad  $C_5$  of Type I-i or I-iii, there exists an edge of  $F_v$  incident with  $x_4$  in  $G$ . Then let  $e_1$  be such an edge and let  $e_2 = x_2x_3$ . Suppose next that  $C_v$  is a bad  $C_5$  of Type I-ii. In this case, we may assume that for all  $1 \leq i \leq 4$ ,  $x_i$  is not incident with an edge in  $M_3$ , and we let  $e_1 = x_1x_2$  and  $e_2 = x_3x_4$ . In either case, note that  $e_1$  and  $e_2$  can be added into  $M_3$  as a matching.

Similarly, we can find two edges  $e'_1$  and  $e'_2$  from  $C_{v'}$  such that  $e'_1$  and  $e'_2$  can be added into  $M_3$  as a matching. Moreover, when we reconstruct  $C_v$  and  $C_{v'}$ , we can add

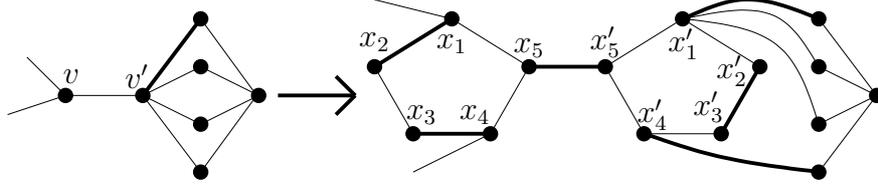


Figure 5: The matching in  $G$ .

the five edges  $e_1, e_2, e'_1, e'_2$  and  $vv'$  into  $M_3$  and obtain a matching in  $G$ . See Figure 5. This implies that for each  $S_v \in \mathcal{S}_1$ , we can add  $2|V(S_v) \cap (V_1^S \cup V_1^N)| + 1$  edges into  $M_3$  through reconstructions at  $C_u$  for all uncovered vertices  $u$  in  $S_v$ , where  $C_u$  is a bad  $C_5$  corresponding to  $u$ . Thus,  $G$  has a matching with at least  $|M_3| + 2|V(K)| + |\mathcal{S}_1|$  edges, that is,

$$\alpha'(G) \geq |M_3| + 2|V(K)| + |\mathcal{S}_1|. \quad (8)$$

Hence by Claim 4 and by the inequalities (3) – (8),

$$\begin{aligned} \alpha'(G) &\geq |M_3| + 2|V(K)| + |\mathcal{S}_1| \\ &\geq \sum_{F \in \mathcal{D}_3} f(F) - 1 + 2|V(K)| + |\mathcal{S}_1| \\ &\geq \frac{3}{2}|\mathcal{D}_3^S| + 2|\mathcal{D}_3^N| + \frac{5}{2}(|\tilde{\mathcal{D}}| - |\mathcal{D}_3^S| - |\mathcal{D}_3^N|) + 2|V(K)| + |\mathcal{S}_1| - 1 \\ &= \frac{5}{2}|\tilde{\mathcal{D}}| - |\mathcal{D}_3^S| - \frac{1}{2}|\mathcal{D}_3^N| + 2(|V_0^S| + |V_1^S| + |V_2^S| + |V_0^N| + |V_1^N| + |V_2^N|) + |\mathcal{S}_1| - 1 \\ &\geq \frac{5}{2}|\tilde{\mathcal{D}}| + |V_0^S| + |V_1^S| + |V_2^S| + \frac{3}{2}(|V_0^N| + |V_1^N| + |V_2^N|) + |\mathcal{S}_1| - 1 \\ &\geq \frac{5}{2}|\tilde{\mathcal{D}}| + \frac{5}{8}(|V_0^S| + |V_0^N|) + \frac{3}{4}(|V_1^S| + |V_1^N|) + |\mathcal{S}_1| + (|V_2^S| + |V_2^N|) + \frac{1}{2}|V_2^N| - 1 \\ &\geq \frac{5}{2}|\tilde{\mathcal{D}}| + \frac{5}{2}l + \frac{3}{2}|\mathcal{S}_1| + |\mathcal{S}_1| + 2|\mathcal{S}_2| + \frac{1}{2}|\mathcal{S}_2| - 1 \\ &= \frac{5}{2}|\mathcal{D}| - 1, \end{aligned}$$

or

$$|\mathcal{D}| \leq \frac{2}{5}(\alpha'(G) + 1).$$

This completes the proof of Theorem 7.  $\square$

## Acknowledgements

The authors would like to thank anonymous referees for careful reading the previous manuscript and helpful suggestions.

## References

- [1] H. Broersma, D. Paulusma and K. Yoshimoto, Sharp upper bounds on the minimum number of components of 2-factors in claw-free graphs, *Graphs and Combin.* **25** (2009) 427–460.
- [2] R. Čada, S. Chiba and K. Yoshimoto, A 2-factor in which each cycle has long length in claw-free graphs, submitted.
- [3] P.A. Catlin, A reduction method to find spanning eulerian subgraphs, *J. Graph Theory* **12** (1988) 29–44.
- [4] S.A. Choudum and M.S. Paulraj, Regular factors in  $K_{1,3}$ -free graphs, *J. Graph Theory* **15** (1991) 259–265.
- [5] Y. Egawa and K. Ota, Regular factors in  $K_{1,n}$ -free graphs, *J. Graph Theory* **15** (1991) 337–344.
- [6] R.J. Faudree, O. Favaron, E. Flandrin, H. Li and Z. Liu, On 2-factors in claw-free graphs, *Discrete Math.* **206** (1999) 131–137.
- [7] R.J. Faudree, E. Flandrin and Z. Ryjáček, Claw-free graphs—a survey, *Discrete Math.* **164** (1997) 87–147.
- [8] R.J. Faudree, C. Magnant, K. Ozeki and K. Yoshimoto, Claw-free graphs and 2-factors that separate independent vertices, *J. Graph Theory* **69** (2012) 251–263.
- [9] R.J. Gould and E. Hynds, A note on cycles in 2-factors of line graphs, *Bull. ICA* **26** (1999) 46–48.
- [10] R.J. Gould and M.S. Jacobson, Two-factors with few cycles in claw-free graphs, *Discrete Math.* **231** (2001) 191–197.
- [11] Z. Hu, F. Tian and B. Wei, Hamilton connectivity of line graphs and claw-free graphs, *J. Graph Theory* **50** (2005) 130–141.
- [12] B. Jackson, Hamilton cycles in 7-connected line graphs, preprint (1989).
- [13] B. Jackson and K. Yoshimoto, Even subgraphs of bridgeless graphs and 2-factors of line graphs, *Discrete Math.* **307** (2007) 2775–2785.
- [14] B. Jackson and K. Yoshimoto, Spanning even subgraphs of 3-edge-connected graphs, *J. Graph Theory* **62** (2009) 37–47.
- [15] T. Kaiser and P. Vrána, Hamilton cycles in 5-connected line graphs, *European J. Combin.* **33** (2012) 924–947.

- [16] R. Kužel, K. Ozeki and K. Yoshimoto, 2-factors and independent sets on claw-free graphs, *Discrete Math.* **312** (2012) 202–206.
- [17] M.M. Matthews and D.P. Sumner, Hamiltonian results in  $K_{1,3}$ -free graphs, *J. Graph Theory* **8** (1984) 139–146.
- [18] Z. Ryjáček, On a closure concept in claw-free graphs, *J. Combin. Theory Ser. B* **70** (1997) 217–224.
- [19] Z. Ryjáček, A. Saito and R.H. Schelp, Closure, 2-factor, and cycle coverings in claw-free graphs, *J. Graph Theory* **32** (1999) 109–117.
- [20] C. Thomassen, Reflections on graph theory, *J. Graph Theory* **10** (1986) 309–324.
- [21] K. Yoshimoto, On the number of components in 2-factors of claw-free graphs *Discrete Math.* **307** (2007) 2808–2819.
- [22] S. Zhan, On hamiltonian line graphs and connectivity, *Discrete Math.* **89** (1991) 89–95.
- [23] M. Zhan, Hamiltonicity of 6-connected line graphs, *Discrete Appl. Math.* **158** (2010) 1971–1975.